Hedging Costs for Variable Annuities under Regime-Switching[*]

P. Azimzadeh [†] P.A. Forsyth [‡] K.R. Vetzal [§]

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Abstract

A general methodology is described in which policyholder behaviour is decoupled from the pricing of a variable annuity based on the cost of hedging it, yielding two weakly coupled systems of partial differential equations (PDEs): the pricing and utility systems. The utility system is used to generate policyholder withdrawal behaviour, which is in turn fed into the pricing system as a means to determine the cost of hedging the contract. This approach allows us to incorporate the effects of utility-based pricing and factors such as taxation. As a case study, we consider the Guaranteed Lifelong Withdrawal and Death Benefits (GLWDB) contract. The pricing and utility systems for the GLWDB are derived under the assumption that the underlying asset follows a Markov regime-switching process. An implicit PDE method is used to solve both systems in tandem. We show that for a large class of utility functions, the pricing and utility systems preserve homogeneity, allowing us to decrease the dimensionality of the PDEs and thus to rapidly generate numerical solutions. It is shown that for a typical contract, the fee required to fund the cost of hedging calculated under the assumption that the policyholder withdraws at the contract rate is an appropriate approximation to the fee calculated assuming optimal consumption. The costly nature of the death benefit is documented. Results are presented which demonstrate the sensitivity of the hedging expense to various parameters.

Keywords: Variable annuity, Guaranteed lifelong withdrawal and death benefits, regime-switching, hedging costs, optimal consumption, utility-based pricing

1 Introduction

Variable annuities are tax-deferred, unit-linked insurance products. These products are a class of insurance vehicles that provide the buyer with particular guarantees without requiring them to sacrifice full control over the funds invested. These funds are usually invested in a collective investment vehicle such as a mutual fund and the writer's position is secured by the deduction of a proportional fee applied to each investors' account.

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[†]Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada N2L 3G1 pazimzad[at]uwaterloo[dot]ca

[‡]Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada N2L 3G1 paforsyt[at]uwaterloo[dot]ca

[§]School of Accounting and Finance, University of Waterloo, Waterloo ON, Canada N2L 3G1 kvetzal[at]uwaterloo[dot]ca

We propose a method for pricing such contracts when the value of the underlying investment follows a Markovian regime-switching process. Regime-switching was introduced by Hamilton (1989), while its application to long-term guarantees was popularized by Hardy (2001), who demonstrated its effectiveness by fitting to the S&P 500 and the Toronto Stock Exchange 300 indices. Regime-switching has thus been suggested as a sensible model for pricing variable annuities (Siu 2005, Lin et al. 2009, Bélanger et al. 2009, Yuen and Yang 2010, Ngai and Sherris 2011, Jin et al. 2011) due to their long-term nature. An alternative to this model is stochastic volatility (Hull and White 1987). However, it could be argued that due to the long-term nature of these guarantees, it is more useful to choose a model which allows for the incorporation of a long-term economic perspective. A regime-switching process has parameters which are economically meaningful, and it is straightforward to adjust these parameters to incorporate economic views. This is perhaps more difficult for a stochastic volatility model, which is typically calibrated to short term option prices. Furthermore, the adoption of stochastic volatility requires an additional dimension in the corresponding partial differential equation (PDE) while the regime-switching model adds complexity proportional to the number of regimes considered, and as a result is computationally less intensive. Moreover, it is straightforward (in the regime-switching framework) to allow for different levels of the risk-free interest rate across regimes. The alternative of incorporating an additional stochastic interest rate factor would add an extra dimension to the PDE, with the associated costs of complexity.

We demonstrate our methodology by considering a specific variable annuity: the Guaranteed Lifelong Withdrawal and Death Benefits (GLWDB) contract. The GLWDB is a response to a general reduction in the availability of defined benefit pension plans, allowing the buyer to replicate the security of such a plan via a substitute. The GLWDB is bootstrapped via a lump sum payment to an insurer, $S(0)$, which is invested in risky assets. We term this the *investment account*. Associated with the GLWDB contract are the *guaranteed withdrawal benefit account* and the *guaranteed death benefit account*, hereafter referred to as the withdrawal and death benefits for brevity. We also refer to these as the *auxiliary accounts*. Both auxiliary accounts are initially set to $S(0)$. At a finite set of *withdrawal dates*, the policyholder is entitled to withdraw a predetermined fraction of the withdrawal benefit (or any lesser amount), even if the investment account diminishes to zero. This predetermined fraction is referred to as the *contract withdrawal rate*. If the policyholder wishes to withdraw in excess of the contract withdrawal rate, they can do so upon the payment of a penalty. Typical GLWDB contracts include penalty rates that are decreasing functions of time. Upon death, the policyholder's estate receives the maximum of the investment account and death benefit. These contracts are often bundled with *ratchets* (a.k.a. step-ups), a contract feature that periodically increases one or more of the auxiliary accounts to the investment account, provided that the investment account has grown larger than the respective auxiliary account. Moreover, *bonus* (a.k.a. roll-up) provisions are also often present, in which the withdrawal benefit is increased if the policyholder does not withdraw on a given withdrawal date.

This contract can be considered as part of a greater family of insurance vehicles offering guaranteed benefits that have emerged as a result of a recent trend away from defined benefits (Butrica et al. 2009). Our approach can easily be extended to include features present in an arbitrary member of this family. There exists a maturing body of work on pricing these contracts. Bauer et al. (2008) introduce a general framework for pricing various products in this family. Monte Carlo and numerical integration are employed, and loss-maximizing (from the perspective of the insurer) withdrawal strategies are considered. Holz et al. (2007) compute the fair fee for Guaranteed Lifelong Withdrawal Benefit (GLWB) contracts via a Monte Carlo method. Milevsky and Salisbury

(2006) employ a numerical PDE approach to price the Guaranteed Minimum Withdrawal Benefits (GMWB) contract. Shah and Bertsimas (2008) introduce a GLWB model with stochastic volatility and consider static strategies. Kling et al. (2011) provide an extension of the variable annuity model under stochastic volatility. Piscopo and Haberman (2011) consider a model with stochastic mortality risk.

In the general area of financial derivatives, the traditional approach is to assume that the policyholder acts so as to maximize the value of owning the contract. The no-arbitrage price of the contract is then calculated as the cost to the writer of the contract of establishing a self-financing hedging strategy that is guaranteed to produce at least enough cash to pay off any future liabilities resulting from the policyholder's future decisions with respect to the contract (in the context of the assumed pricing model). Since derivative payoffs are a zero sum game, this is equivalent to establishing a price on the basis of assuming a worst case scenario to the contract writer. We will refer to the assumption of such behaviour by policyholders here as *loss-maximizing* strategies, as they represent worst case outcomes for the insurer. Such strategies produce an upper bound on the fair price of the contract, but it is far from clear that policyholders actually behave in this manner. Instead, for any of a number of reasons, a policyholder may deviate from loss-maximizing behaviour.

In order to account for this, we provide a new approach here in which we decouple policyholder withdrawal behaviour from the contract pricing equations, and generate said behaviour by considering a policyholder's utility. This general approach is applicable to any contract involving policyholder behaviour, and results in two weakly coupled systems of PDEs. In the context of GLWDBs, this allows for the easy modeling of complex phenomena such as risk aversion and taxation. Solving the PDEs backwards in time allows us to employ the Bellman principle to ensure that the policyholder is able to maximize his or her utility. Since our approach incorporates this added generality, we will generally avoid the use of the term "no-arbitrage" below, and instead refer to the cost of hedging. Of course, under the specific case of loss-maximizing behaviour by the policyholder, our cost of hedging coincides with the traditional no-arbitrage price.

In §2, we introduce a system of regime-switching PDEs used to determine the hedging costs of the GLWDB contract. In §3, we introduce a system of regime-switching PDEs used to model a policyholder's utility and describe how this system is used alongside the system introduced in §2 to determine the cost of hedging the guarantee assuming optimal consumption. In §4, we discuss our numerical methodology. In §5, we present results under both the assumption that the policyholder behaves so as to maximize the cost of the guarantee (i.e. the loss-maximizing strategy) and the assumption that the policyholder maximizes utility.

Overall, the contributions of this work are:

- We introduce a general methodology that allows for the decoupling of policyholder behaviour from the cost of hedging the contract.
  - This approach yields two weakly coupled systems of PDEs: the pricing and utility systems.
  - This approach abandons the arguably flawed notion of a policyholder acting only so as to maximize the cost of a guarantee.
- We model the long-term behaviour of the underlying stock index (or mutual fund) by a Markovian regime-switching process.

- We present the pricing and utility systems for the GLWDB contract.
- We show sufficient conditions for the homogeneity of the systems. This result is computationally significant, as it is used to reduce the dimensionality of the systems.
- We find that assuming optimal consumption yields a hedging cost fee that is very close to the fee calculated by assuming that the policyholder follows the static strategy of always withdrawing at the contract rate. This is a result of particular practical importance as it suggests that policyholders will generally withdraw at the contract rate. This substantiates pricing contracts under this otherwise seemingly naïve assumption.
- We find that the inclusion of a death benefit is often expensive. This may account for the failure to properly hedge this guarantee and the subsequent withdrawal of contracts including ratcheting death benefits from the Canadian market.
- We demonstrate sensitivity to various parameters and we consider the adoption of exotic fee structures in which the proportional fee applies not just to the investment account but rather to the greater of this account and one or more of the auxiliary accounts.

2 Hedging costs

We begin by considering a basic model for pricing GLWDBs under which policyholder withdrawal behaviour is determined so as to maximize the value of the guarantee (i.e. the loss-maximizing strategy). We extend previous work by Forsyth and Vetzal (2013) via the introduction of a death benefit. For simplicity, we first consider the single-regime case and subsequently extend this model to include regime-switching.

2.1 Derivation of the pricing equation

Let $\mathcal{M}(t)$ be defined as the instantaneous rate of mortality per unit interval. The fraction of policyholders still alive at time t is

$$\mathcal{R}(t) = 1 - \int_0^t \mathcal{M}(s)\, ds,$$

where $t = 0$ is the time at which the contract is purchased. Let $S(t)$ be the amount in the investment account of any policyholder of the GLWDB contract who is still alive at time t. Let $W(t)$ and $D(t)$ be the withdrawal and death benefits at time t. Assume that the underlying value of the investment account is described by

$$dS = (\mu - \alpha) S dt + \sigma S dZ$$

where Z is a Wiener process. The constant α represents the total fee structure of the contract. It is comprised of two terms. First, the underlying investment fund has a proportional management fee α_M. Second, the insurer charges for the cost of hedging the contractual features through a proportional fee α_R, which we will refer below to as the hedging cost fee. The total proportional deduction applied to the investor's account is $\alpha = \alpha_M + \alpha_R$. If we suppose that α_M is fixed, the pricing problem becomes one of finding α_R such that the insurer can follow a hedging strategy which (in principle) can eliminate risk. This will be discussed further in §4.3. S tracks the index $\hat{S}$ which follows

$$d\hat{S} = \mu\hat{S}dt + \sigma\hat{S}dZ.$$

It is assumed that the insurer is unable to short S for fiduciary reasons.

We proceed by a hedging argument ubiquitous in the literature (Windcliff et al. 2001, Chen et al. 2008, Bélanger et al. 2009). Let $U(S, W, D, t)$ be the cost of funding the withdrawal and death benefits at time t years after purchase for investment account value S, withdrawal benefit W, and death benefit D. The value of U is adjusted to account for the effects of mortality. We assume that this contract was purchased at time zero by a buyer aged x_0. Let T be the smallest time at which $\mathcal{R}(T) = 0$ (we assume that such a time exists; i.e. no policyholder lives forever). The insurer has no obligations at time T and hence

$$U(S, W, D, T) = 0. \tag{2.1}$$

The writer creates a replicating portfolio Π by shorting one contract and taking a position of x units in the index $\hat{S}$. That is,

$$\Pi(S, W, D, t) = -U(S, W, D, t) + x\hat{S}.$$

The contractually specified times at which withdrawals and ratchets occur are referred to as *event times*, gathered in the set $\mathcal{T} = \{t_1, t_2, \ldots, t_{N-1}\}$ and ordered by

$$0 = t_1 < t_2 < \ldots < t_{N-1} < t_N = T.$$

Note that time zero (but not $t_N = T$) is also referred to as an event time even if no withdrawals or ratchets are prescribed to occur at time zero.

Following standard portfolio dynamics arguments (see, e.g. Forsyth and Vetzal 2013) and noting that between event times, dU is a function solely of S and t, we can use Itô's lemma to yield

$$d\Pi = -\left[\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + (\mu - \alpha) S \frac{\partial U}{\partial S} + \frac{\partial U}{\partial t}\right) dt + \sigma S \frac{\partial U}{\partial S} dZ\right] + x\left[\mu\hat{S}dt + \sigma\hat{S}dZ\right] + \mathcal{R}(t)\,\alpha_R S dt - \mathcal{M}(t)\left[0 \vee (D - S)\right] dt,$$

where $a \vee b = \max(a, b)$. The term $\mathcal{R}(t)\,\alpha_R S dt$ represents the fees collected by the hedger, while $\mathcal{M}(t)\left[0 \vee (D - S)\right] dt$ represents the surplus generated by the death benefit as paid out to the estates of deceased policyholders. Taking $x = \left(S/\hat{S}\right) \frac{\partial U}{\partial S}$ yields

$$d\Pi = \left(-\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \alpha S \frac{\partial U}{\partial S} - \frac{\partial U}{\partial t} + \mathcal{R}(t)\,\alpha_R S - \mathcal{M}(t)\left[0 \vee (D - S)\right]\right) dt. \tag{2.2}$$

As this increment is deterministic, by the principle of no-arbitrage, the corresponding portfolio process must grow at the risk-free rate. That is,

$$d\Pi = r\Pi dt = r\left(-U + \frac{S}{\hat{S}}\frac{\partial U}{\partial S}\hat{S}\right) dt. \tag{2.3}$$

Substituting (2.3) into (2.2),

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + (r-\alpha) S \frac{\partial U}{\partial S} + \frac{\partial U}{\partial t} - rU - \mathcal{R}(t)\,\alpha_R S + \mathcal{M}(t)\left[0 \vee (D-S)\right] = 0. \tag{2.4}$$

Let

$$V(S,W,D,t) = U(S,W,D,t) + \mathcal{R}(t)\, S \tag{2.5}$$

be the cost of funding the entire contract at time t. Substituting into (2.4), we arrive at

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\alpha) S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV + \mathcal{R}(t)\,\alpha_M S + \mathcal{M}(t)(S \vee D) = 0. \tag{2.6}$$

We stress that V satisfies the above PDE only *between* a pair of adjacent event times t_n and t_{n+1}. We discuss the behaviour of V *across* event times (e.g. from t_n^- to t_n^+) in §2.2.

2.2 Events

Remark 2.1 (Notation)**.** *In order to reduce clutter, we will sometimes refer to* $V(S,W,D,t)$ *as* $V(\mathbf{x},t)$*, where* $\mathbf{x}=(S,W,D)$*. We will often use this notation for other functions of* (S,W,D) *as well. We refer to a point* $\mathbf{x}$ *as a* state*.*

Event times. Across event times, V is not necessarily continuous as a function of t. We restrict V to be a *càglàd* function of t so that for all $\mathbf{x}$, $V(\mathbf{x},t)=\lim_{s\uparrow t} V(\mathbf{x},s)$ and $V(\mathbf{x},t^+)=\lim_{s\downarrow t} V(\mathbf{x},s)$ exist. Whenever $t\in\mathcal{T}$, $V(\mathbf{x},t)$ and $V(\mathbf{x},t^+)$ can be regarded as the price of the contract "immediately before" and "immediately after" the event time, respectively.

Withdrawal strategy. We isolate the withdrawal strategy by introducing a function $\gamma(\mathbf{x},t)$ describing the policyholder's actions at state $\mathbf{x}$ and $t\in\mathcal{T}$.

- $\gamma(\mathbf{x},t)=0$ indicates that the policyholder does not withdraw anything.
- $\gamma(\mathbf{x},t)\in(0,1]$ indicates a nonzero withdrawal less than or equal to the *contract withdrawal amount*, the maximum amount one can withdraw without incurring a penalty.
- $\gamma(\mathbf{x},t)\in(1,2]$ indicates withdrawal at more than the contract withdrawal amount.

$\gamma(\mathbf{x},t)=2$ is referred to as a *full surrender*, as it corresponds to the scenario in which the policyholder withdraws the entirety of their investment account, while $\gamma(\mathbf{x},t)\in(1,2)$ is referred to as a *partial surrender*.

Remark 2.2 (Abstract strategy)**.** *We stress that we have not yet made any assumptions about policyholder behaviour. The decoupling of policyholder behaviour from the hedging cost equations is the guiding philosophy of this work, and allows us to model complex phenomena visible to the policyholder, but not necessarily visible to the writer. To be more precise, we assume that the insurer can observe the policyholder's strategy, though not the factors which determine that strategy. The robustness of this approach is made concrete via the model developed in §3, which considers the effects of taxation and nonlinear utility functions on a policyholder's withdrawal strategy.*

Denote the cost of funding the contract at state $\mathbf{x}$ and event time $t \in \mathcal{T}$ assuming the policyholder performs action $\lambda \in [0,2]$ by

$$v(\mathbf{x}, t, \lambda) = V\left(\mathbf{f}(\mathbf{x}, t, \lambda), t^{+}\right) + \mathcal{R}(t) f(\mathbf{x}, t, \lambda) \tag{2.7}$$

where f represents cash flow from the writer to the policyholder and $\mathbf{f}\colon \mathbb{R}^3 \times \mathcal{T} \times [0,2] \to \mathbb{R}^3$ describes the state of the contract after the event. The cash flow is adjusted to account only for the fraction of holders still alive at time t, $\mathcal{R}(t)$. The actual (observed) cost of funding the contract is obtained simply by passing the withdrawal strategy employed by the policyholder γ to v. That is,

$$V(\mathbf{x}, t) = v(\mathbf{x}, t, \gamma(\mathbf{x}, t)). \tag{2.8}$$

We cast a withdrawal event in the form (2.7) by considering the three cases enumerated above (i.e. $\lambda = 0$, $\lambda \in (0,1]$, and $\lambda \in (1,2]$) separately.

In the following, we refer to $\mathcal{T}_{\text{Withdraw}} \subset \mathcal{T}$ as the set times at which withdrawals are prescribed and $\mathcal{T}_{\text{Ratchet}} \subset \mathcal{T}$ as the set of times at which ratchets are prescribed. We begin by assuming $\mathcal{T}_{\text{Withdraw}} \bigcap \mathcal{T}_{\text{Ratchet}} = \emptyset$ (i.e. ratchets and withdrawals do not occur simultaneously) and subsequently relax this assumption.

Bonus. At a time $t \in \mathcal{T}_{\text{Withdraw}}$, nonwithdrawal is indicated by $\lambda = 0$. If the policyholder chooses not to withdraw, the withdrawal benefit is amplified by $1 + B(t)$, where $B(t)$ is the bonus rate available at t. By the principle of no-arbitrage,

$$v(S, W, D, t, 0) = V\left(\underbrace{S,\ W(1 + B(t)),\ D}_{\mathbf{f}(\mathbf{x},t,0)},\ t^{+}\right).$$

Withdrawal not exceeding the contract rate. At a time $t \in \mathcal{T}_{\text{Withdraw}}$, the contract withdrawal amount for withdrawal benefit W is $G(t)W$. $G(t)$, the *contract withdrawal rate* at time t, is specified by the contract. The amount withdrawn by the policyholder when $\lambda \in (0,1]$ is $\lambda G(t) W$. We express this type of withdrawal as

$$v(S, W, D, t, \lambda) = V\left(\underbrace{(S - \lambda G(t) W) \vee 0,\ W,\ (D - \lambda G(t) W) \vee 0}_{\mathbf{f}(\mathbf{x},t,\lambda)},\ t^{+}\right) + \mathcal{R}(t) \underbrace{\lambda G(t) W}_{f(\mathbf{x},t,\lambda)}.$$

For the particular contract that we are considering, the death benefit is reduced whenever any withdrawals are made.

Partial or full surrender. At a time $t \in \mathcal{T}_{\text{Withdraw}}$, The amount withdrawn if $\lambda \in (1,2]$ is

$$G(t) W + (\lambda - 1)(1 - \kappa(t)) S'$$

where $S' = (S - G(t) W) \vee 0$ is the state of the investment account after a withdrawal at the contract withdrawal amount and $\kappa(t) \in [0,1]$ is the *penalty rate* incurred at t for withdrawing above the contract withdrawal amount. For a typical contract, $\kappa(t)$ is monotonically decreasing in time. We express this type of withdrawal as

$$v(S,W,D,t,\lambda) = V\Bigg(\underbrace{(2-\lambda)\,S',\ (2-\lambda)\,W,\ (2-\lambda)\,D}_{\mathbf{f}(\mathbf{x},t,\lambda)},\ t^{+}\Bigg) + \mathcal{R}(t)\underbrace{\left(G(t)\,W + (\lambda-1)\,(1-\kappa(t))\,S'\right)}_{f(\mathbf{x},t,\lambda)}.$$

Ratchets. At a time $t \in \mathcal{T}_{\text{Ratchet}}$, the withdrawal benefit is increased to the investment account if the latter has grown larger than the former in value. Note that the value of the withdrawal benefit W can never decrease, unless a penalty has been incurred for withdrawing over the contract withdrawal rate. Although ratchets are not controlled by the policyholder, we can still write a ratchet event in the form (2.7) by

$$v(S,W,D,t,\lambda) = V\Bigg(\underbrace{S,\ S\vee W,\ D}_{\mathbf{f}(\mathbf{x},t,\lambda)},\ t^{+}\Bigg)$$

irrespective of the value of λ. We also explore the possibility of a ratcheting death benefit.

Simultaneous events. When multiple events are prescribed to occur at the same time, we simply apply them one after the other. Naturally, without a particular order, the pricing problem is not well-posed: the contract is ambiguous. If a withdrawal and a ratchet are prescribed to occur at the same time, we assume that the withdrawal occurs before the ratchet. As we are solving the PDE backwards in time in order to employ the Bellman principle, these events are applied in reverse order (in backwards time).

2.3 Loss-maximizing strategies

For all states $\mathbf{x}$ and event times $t \in \mathcal{T}$, let

$$\Gamma(\mathbf{x},t) = \operatorname*{arg\,max}_{\lambda\in[0,2]}\left[v(\mathbf{x},t,\lambda)\right] \tag{2.9}$$

Since we are maximizing (2.7), $\Gamma(\mathbf{x},t)$ is simply the set of all actions that maximize the cost of the contract at $\mathbf{x}$ and t. If the writer is interested in computing the hedging cost for the contract in the worst-case scenario, the withdrawal strategy is assumed to satisfy

$$\gamma(\mathbf{x},t) \in \Gamma(\mathbf{x},t) \tag{2.10}$$

for all $\mathbf{x}$ and $t \in \mathcal{T}$. Any such strategy is termed a *loss-maximizing withdrawal strategy.*

Remark 2.3 (An unfortunate choice of terms)**.** *A loss-maximizing withdrawal strategy is often referred to as an optimal strategy in the literature. The adoption of the term optimal is an arguably unfortunate one, as an optimal strategy is not necessarily "optimal" for the policyholder. We stress that an optimal strategy as typically referred to in the literature is simply one that maximizes losses for the writer, and use instead the term "loss-maximizing" for the remainder of this work in order to avoid confusion.*

2.4 Regime-switching

We extend the formulation to include a regime-switching framework in which shifts between states are controlled by a continuous-time Markov chain. Letting $\mathcal{S} = \{1, 2, \ldots, M\}$ be the state-space consisting of M regimes, we assume that in regime $i \in \mathcal{S}$, the underlying investment account evolves according to

$$dS = (\mu_i - \alpha) S + \sigma_i S dZ + \sum_{j=1}^{M} S (J_{i\to j} - 1) dX_{i\to j}$$

where

$$dX_{i\to j} = \begin{cases} 1 & \text{with probability } \delta_{i,j} + q_{i\to j} dt \\ 0 & \text{with probability } 1 - (\delta_{i,j} + q_{i\to j} dt) \end{cases}$$

and $\delta_{i,j}$ is the Kronecker delta. Here, $q_{i\to j}$ is the objective ($\mathbb{P}$ measure) rate of transition from regime i to j whenever $i \neq j$ and

$$q_{i\to i} = -\sum_{\substack{j=1 \\ j\neq i}}^{M} q_{i\to j}.$$

$J_{i\to j} \geqslant 0$ is the relative jump size in S associated with a transition from regime i to j. We take $J_{i\to i} = 1$ for all i so that jumps in the underlying are not experienced unless there is a change in regime. Let $V_i(S, W, D, t)$ be the cost of funding a GLWDB in regime i. Following a combination of the hedging arguments in §2.1 and §A, we arrive at the system of PDEs

$$\mathcal{L}_i V_i + \sum_{\substack{j=1 \\ j\neq i}}^{M} \left[q^{\mathbb{Q}}_{i\to j} V_j (J_{i\to j} S, W, D, t) \right] + \frac{\partial V_i}{\partial t} + \mathcal{R}(t) \alpha_M S + \mathcal{M}(t) (S \vee D) = 0 \ \forall i \in \mathcal{S} \tag{2.11}$$

where

$$\mathcal{L}_i = \frac{1}{2} \sigma_i^2 S^2 \frac{\partial^2}{\partial S^2} + \left(r_i - \alpha - \rho^{\mathbb{Q}}_i \right) S \frac{\partial}{\partial S} - \left(r_i - q^{\mathbb{Q}}_{i\to i} \right).$$

$q^{\mathbb{Q}}_{i\to j}$ is the risk-neutral rate of transition from regime i to j whenever $i \neq j$ and

$$q^{\mathbb{Q}}_{i\to i} = -\sum_{\substack{j=1 \\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j}.$$

Furthermore, $\rho^{\mathbb{Q}}_i$ is defined as

$$\rho^{\mathbb{Q}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{M} \left[q^{\mathbb{Q}}_{i\to j} (J_{i\to j} - 1) \right] = \sum_{j=1}^{M} \left[q^{\mathbb{Q}}_{i\to j} J_{i\to j} \right].$$

(2.11) is referred to as the *pricing system*.

The events introduced in the single-regime model are simply applied to each regime separately. That is, the regime-switching analogues of (2.7) and (2.8) are

$$v_i (\mathbf{x}, t, \lambda) = V_i \left(\mathbf{f} (\mathbf{x}, t, \lambda), t^+ \right) + \mathcal{R}(t) f (\mathbf{x}, t, \lambda) \tag{2.12}$$

and

$$V_i(\mathbf{x},t) = v_i(\mathbf{x},t,\gamma_i(\mathbf{x},t)). \tag{2.13}$$

Likewise, the withdrawal strategy becomes regime-dependent. The regime-switching analogue of (2.9) and (2.10) is

$$\gamma_i(\mathbf{x},t) \in \Gamma_i(\mathbf{x},t) = \arg\max_{\lambda\in[0,2]} [v_i(\mathbf{x},t,\lambda)]. \tag{2.14}$$

3 Optimal consumption

Using a loss-maximizing strategy yields the largest hedging cost fee. Any other strategy will, by definition, yield a smaller fee. Using the fee generated by a loss-maximizing strategy ensures that the writer can, at least in theory, hedge a short position in the contract with no risk. However, insurers are often interested in using a less conservative method for pricing contracts so as to decrease the hedging cost fee while minimizing their exposure. We now extend the framework introduced in §2 to strategies based on optimal consumption from the perspective of the policyholder. As usual, we first consider the single-regime case and subsequently provide the extension to include regime-switching.

3.1 Utility PDE

Let $\overline{V}(S,W,D,t)$ be the mortality-adjusted utility of holding a GLWDB contract at t with investment account value S, withdrawal benefit W and death benefit D. Following standard arguments, we express the evolution of a policyholder's utility by

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \overline{V}}{\partial S^2} + (\mu-\alpha) S \frac{\partial \overline{V}}{\partial S} + \frac{\partial \overline{V}}{\partial t} - \beta \overline{V} + \mathcal{M}(t)\, u^B(S \vee D) = 0. \tag{3.1}$$

Here, $u^B(x)$ is the *bequest utility*, the utility received from bequeathing x, and β is the *rate of time preference*. Note that (3.1) depends on the real-world drift μ as opposed to the risk-free rate r. We represent the worthlessness of holding a GLWDB after all death benefits have been paid by

$$\overline{V}(S,W,D,T) = 0. \tag{3.2}$$

The drift-diffusion form (3.1) corresponds to a standard additive utility specification.

3.2 Events

As in (2.7) and (2.8), we parameterize an event occurring at $t \in \mathcal{T}$ by writing it in the form

$$\overline{v}(\mathbf{x},t,\lambda) = \overline{V}\left(\mathbf{f}(\mathbf{x},t,\lambda), t^+\right) + \mathcal{R}(t)\, \overline{f}(\mathbf{x},t,\lambda) \tag{3.3}$$

along with

$$\overline{V}(\mathbf{x},t) = \overline{v}(\mathbf{x},t,\gamma(\mathbf{x},t)). \tag{3.4}$$

$\mathbf{f}$ is defined implicitly for each event type in §2.2. It should be noted that the function $\overline{f}$ does not represent a cash flow, but rather an influx of utility to the holder. That is,

$$\overline{f}(\mathbf{x},t,\lambda) = u^C(f(\mathbf{x},t,\lambda)),$$

where f is defined for each event type in §2.2 and $u^C(y)$ is the *consumption utility*, the utility received from consuming y.

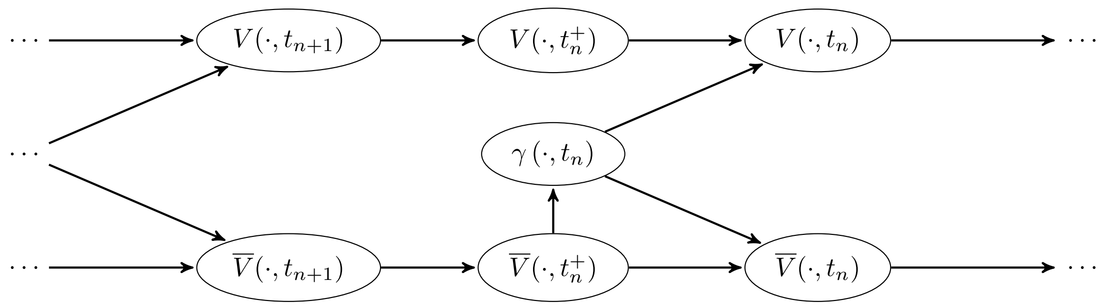


FIGURE 3.1: *A graph depicting the propagation of information in the pricing procedure.*

3.3 Consumption-optimal withdrawal

We refer to a withdrawal strategy that satisfies

$$\gamma(\mathbf{x}, t) \in \overline{\Gamma}(\mathbf{x}, t) = \underset{\lambda \in [0,2]}{\arg\max} \left[\overline{v}(\mathbf{x}, t, \lambda) \right] \tag{3.5}$$

(for all states $\mathbf{x}$ and event times $t \in \mathcal{T}$) as a *consumption-optimal withdrawal strategy*. Since we are maximizing (3.3), $\overline{\Gamma}(\mathbf{x}, t)$ is simply the set of all actions that maximize the policyholder's utility at $\mathbf{x}$ and t.

It should be noted that we are not interested in the value of the numerical solution to the utility PDE but rather in the withdrawal strategy generated by it. Instead of adopting the optimal withdrawal strategy introduced in §2.3, we feed the withdrawal strategy generated by the policyholder's utility into the pricing problem. Given the Cauchy data $V(\cdot, t_{n+1})$ and $\overline{V}(\cdot, t_{n+1})$:

1. Solve $V(\cdot, t_n^+)$ using (2.6) and Cauchy data $V(\cdot, t_{n+1})$.
2. Solve $\overline{V}(\cdot, t_n^+)$ using (3.1) and Cauchy data $\overline{V}(\cdot, t_{n+1})$.
3. Determine $\gamma(\cdot, t_n)$ s.t. (3.5) is satisfied. In doing so, determine $\overline{V}(\cdot, t_n)$ by (3.3) and (3.4).
4. Use $\gamma(\cdot, t_n)$, (2.7) and (2.8) to determine $V(\cdot, t_n)$.

The propagation of information in this procedure is depicted in Figure 3.1.

Remark 3.1 (Ensuring uniqueness)**.** *Step 3 requires that for each $\mathbf{x}$, we determine $\gamma(\mathbf{x}, t_n)$. The expression (3.5) suggests that $\gamma(\mathbf{x}, t_n)$ need not be unique. To ensure the uniqueness of V, we need a way to break ties between consumption-optimal strategies. Formally, we substitute condition (3.5) for*

$$\gamma(\mathbf{x}, t) = c\left(\overline{\Gamma}(\mathbf{x}, t)\right)$$

where c is a choice function on the power set of $[0, 2]$. For example, a choice function c that selects the smallest element (e.g. $c(\{0, 1, 2\}) = 0$) corresponds to a policyholder who will always withdraw the least amount possible to break a tie.

3.4 Regime-switching

Assuming the regime-switching model introduced in §2.4, define $\overline{V}_i(S, W, D, t)$ as the mortality-adjusted utility of holding a GLWDB contract at time t years after purchase in regime $i \in \mathcal{S}$. Following standard arguments, we arrive at

$$\frac{\partial \overline{V}_i}{\partial t} + \overline{\mathcal{L}}_i \overline{V}_i + \sum_{\substack{j=1 \\ j \neq i}}^{M} \left[q_{i \to j} \overline{V}_j \left(J_{i \to j} S, W, D, t \right) \right] + \mathcal{M}(t)\, u_i^B (S \vee D) = 0 \ \forall i \in S \tag{3.6}$$

where

$$\overline{\mathcal{L}}_i = \frac{1}{2} \sigma_i^2 S^2 \frac{\partial^2}{\partial S^2} + (\mu_i - \alpha)\, S \frac{\partial}{\partial S} - (\beta_i - q_{i \to i}).$$

(3.6) is referred to as the *utility system*. Note that this system of PDEs does not depend on the risk-neutral rates of transition $q_{i \to j}^{\mathbb{Q}}$ as in §2.4, but instead on the objective ($\mathbb{P}$ measure) rates of transition $q_{i \to j}$. We use the symbols u_i^B and u_i^C to stress that the utility functions can, in general, be regime-dependent.

As in §2.4, events and the corresponding withdrawal strategies become regime-dependent. The regime-switching analogue of (3.3) and (3.4) is

$$\overline{v}_i(\mathbf{x}, t, \lambda) = \overline{V}_i \left(\mathbf{f}(\mathbf{x}, t, \lambda)\, t^+ \right) + \mathcal{R}(t)\, u_i^C \left(f(\mathbf{x}, t, \lambda) \right) \tag{3.7}$$

and

$$\overline{V}_i(\mathbf{x}, t) = \overline{v}_i(\mathbf{x}, t, \gamma_i(\mathbf{x}, t)). \tag{3.8}$$

Likewise, the regime-switching analogue of (3.5) is

$$\gamma_i(\mathbf{x}, t) \in \overline{\Gamma}_i(\mathbf{x}, t) = \operatorname*{arg\,max}_{\lambda \in [0,2]} \left[\overline{v}_i(\mathbf{x}, t, \lambda) \right]. \tag{3.9}$$

In this way, at any event time, the policyholder's utility in regime i (i.e. $\overline{V}_i$) is directly related to the price in regime i (i.e. V_i). In particular, the algorithm presented in §3.3 becomes:

1. Solve $\langle V_1(\cdot, t_n^+), \ldots, V_M(\cdot, t_n^+) \rangle$ using (2.11) and Cauchy data $\langle V_1(\cdot, t_{n+1}), \ldots, V_M(\cdot, t_{n+1}) \rangle$.
2. Solve $\left\langle \overline{V}_1(\cdot, t_n^+), \ldots, \overline{V}_M(\cdot, t_n^+) \right\rangle$ using (3.6) and Cauchy data $\left\langle \overline{V}_1(\cdot, t_{n+1}), \ldots, \overline{V}_M(\cdot, t_{n+1}) \right\rangle$.
3. For each regime i,
   (a) Determine $\gamma_i(\cdot, t_n)$ such that (3.9) is satisfied. In doing so, determine $\overline{V}_i(\cdot, t_n)$ by (3.7) and (3.8).
   (b) Use $\gamma_i(\cdot, t_n)$, (2.12) and (2.13) to determine $V_i(\cdot, t_n)$.

3.5 Hyperbolic absolute risk-aversion

We consider policyholder consumption to be governed by hyperbolic absolute risk-aversion (HARA) utility (Merton 1970):

$$u_i^C(y; a_i, b_i, p_i) = \lim_{p \to p_i} \frac{1-p}{p} \left(\frac{a_i y}{1-p} + b_i \right)^p. \tag{3.10}$$

We take $u_i^B(y) = h_i u_i^C(y)$, where h_i is termed the *bequest motive*. This is a fairly flexible and general class of utility functions that can be parameterized so that marginal utility is finite at a consumption level of zero. This is potentially of interest in our context since it allows for the possibility that the policyholder will decide to not withdraw any amount at a withdrawal date. Otherwise, with infinite marginal utility at a consumption level of zero, the policyholder will always withdraw some positive amount.

4 Numerical method

4.1 Homogeneity

Let $\mathbf{V}$ denote the column vector consisting of $V_1, V_2, \ldots, V_M$. We define $\overline{\mathbf{V}}$ similarly.

Remark 4.1 (Technical assumptions)**.** *We assume that all regime-switching jumps are unity (i.e. $J_{i\to j} = 1$ for all i and j), that $\mathbf{V}$ (resp. $\overline{\mathbf{V}}$) is a classical solution (i.e. twice differentiable in the investment account S and once in t on (t_n, t_{n+1}) for all $1 \leqslant n < N$) satisfying a growth condition to ensure uniqueness (recall that parabolic PDEs do not, in general, admit unique solutions (Friedman 1964)) and that the functions σ_i, r_i, α, $q^{\mathbb{Q}}_{i\to j}$, μ_i, β_i and $q_{i\to j}$ are bounded and continuous. Under these assumptions, it is possible to use the parametrix method (Levi 1907) to construct a Green's function (denoted F) representation for $\mathbf{V}$ (resp. $\overline{\mathbf{V}}$) on $t \in (t_n, t_{n+1}]$. A more detailed list of these assumptions is provided by Azimzadeh (2013). We further assume that the functions σ_i, r_i, α, and $q^{\mathbb{Q}}_{i\to j}$, μ_i, β_i and $q_{i\to j}$ are independent of S, W and D and exploit this fact in Lemma 4.4.*

Definition 4.2 (Homogeneous function)**.** *A function $s : X \to Y$ between two cones is said to be homogeneous of order $k \in \mathbb{Z}$ if for all $\eta > 0$ and $\mathbf{x} \in X$, $\eta^k s(\mathbf{x}) = s(\eta \mathbf{x})$. We say $\mathbf{V}$ is homogeneous if for each $i \in \mathcal{S}$, V_i is homogeneous.*

Theorem 4.3 (Price homogeneity under loss-maximizing strategy)**.** *Suppose that a loss-maximizing strategy is employed by the policyholder. Then, $\mathbf{V}(\mathbf{x}, t)$ is homogeneous of order 1 in $\mathbf{x}$.*

This fact is established via a series of lemmas. Namely, we show that if $\mathbf{V}(\mathbf{x}, t_{n+1})$ is homogeneous in $\mathbf{x}$, so too is $\mathbf{V}(\mathbf{x}, t_n^+)$ (Lemma 4.4). That is, the system (2.11) composed of the operators $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_M$ preserves homogeneity. Then, we show that if $\mathbf{V}(\mathbf{x}, t_n^+)$ is homogeneous in $\mathbf{x}$, so too is $\mathbf{V}(\mathbf{x}, t_n)$ (Lemma 4.6). That is, homogeneity is preserved across event times under a loss-maximizing strategy. By (2.1) and (2.5), we have $\mathbf{V}(\mathbf{x}, t_N = T) = \mathbf{0}$. Since this is trivially homogeneous, the desired result follows by induction.

Lemma 4.4 (Pricing system homogeneity between event times)**.** *Suppose that for some n with $1 \leqslant n < N$, $\mathbf{V}(\mathbf{x}, t_{n+1})$ is homogeneous of order 1 in $\mathbf{x}$. Then, for all $t \in (t_n, t_{n+1}]$, $\mathbf{V}(\mathbf{x}, t)$ is homogeneous of order 1 in $\mathbf{x}$.*

Proof sketch. If we let $\tau = t_{n+1} - t$ and

$$g(S, W, D, t) = \mathcal{R}(t)\, \alpha_M S + \mathcal{M}(t)(S \vee D),$$

we can write (Remark 4.1)

$$\mathbf{V}(S, W, D, t) = \int_0^\infty F\left(\log \frac{S'}{S}, \tau, 0\right) \mathbf{V}(S', W, D, t_{n+1}) \frac{1}{S'} dS' + \int_0^\tau \int_0^\infty F\left(\log \frac{S'}{S}, \tau, \tau'\right) \left(g(S', W, D, t_{n+1} - \tau')\, \mathbf{1}\right) \frac{1}{S'} dS' d\tau'.$$

where $\mathbf{1}$ is a column vector of ones. The fact that F depends on S' and S only through $\log(S'/S)$ is discussed by Azimzadeh (2013) and stems from the assumption that σ_i, r_i, α and $q^{\mathbb{Q}}_{i \to j}$ are independent of S, W and D (Remark 4.1). The substitution $S' = SS''$ yields

$$\mathbf{V}(S, W, D, t) = \int_0^\infty F\left(\log S'', \tau, 0\right) \mathbf{V}\left(SS'', W, D, t_{n+1}\right) \frac{1}{S''} dS'' + \int_0^\tau \int_0^\infty F\left(\log S'', \tau, \tau'\right) \left(g\left(SS'', W, D, t_{n+1} - \tau'\right) \mathbf{1}\right) \frac{1}{S''} dS'' d\tau'.$$

Since $\mathbf{V}(\mathbf{x}, t_{n+1})$ and $g(\mathbf{x}, t)$ are both homogeneous in $\mathbf{x}$, it is now straightforward to extend $\mathbf{V}(\mathbf{x}, t)$'s homogeneity to $t \in (t_n, t_{n+1}]$.

Remark 4.5. *(Unit jump size assumption) The assumption that the jump sizes are unity $J_{i \to j} = 1$ is required in order to use the standard Green's function form. However, Lemma 4.4 also holds for the case of non-unit jump sizes, but the proof is somewhat more lengthy.*

Lemma 4.6 (Loss-maximizing strategy preserves homogeneity)**.** *Suppose that for some regime $i \in \mathcal{S}$ and for some n with $1 \leqslant n < N$, $V_i(\mathbf{x}, t_n^+)$ is homogeneous of order 1 in $\mathbf{x}$ and that the policyholder employs a loss-maximizing strategy $\gamma_i(\cdot, t_n)$. Then, $V_i(\mathbf{x}, t_n)$ is homogeneous of order 1 in $\mathbf{x}$.*

Proof. We leave it to the interested reader to show that $\mathbf{f}(\mathbf{x}, t_n, \lambda)$ and $f(\mathbf{x}, t_n, \lambda)$ defined implicitly in §2.2 are homogeneous of order 1 in $\mathbf{x}$. From this and the presumed homogeneity of $V_i(\mathbf{x}, t_n^+)$, it follows that $v_i(\mathbf{x}, t_n, \lambda)$ defined by (2.12) is homogeneous of order 1 in $\mathbf{x}$. Let $\eta > 0$ and $\mathbf{x}$ be arbitrary. By (2.14),

$$\begin{aligned}
\gamma_i(\eta\mathbf{x}, t_n) &\in \Gamma_i(\eta\mathbf{x}, t_n) \\
&= \arg\max_{\lambda \in [0,2]} \left[v_i(\eta\mathbf{x}, t_n, \lambda)\right] \\
&= \arg\max_{\lambda \in [0,2]} \eta \left[v_i(\mathbf{x}, t_n, \lambda)\right] \\
&= \arg\max_{\lambda \in [0,2]} \left[v_i(\mathbf{x}, t_n, \lambda)\right] \\
&= \Gamma_i(\mathbf{x}, t_n) \ni \gamma_i(\mathbf{x}, t_n).
\end{aligned}$$

From this, it follows that $v_i(\mathbf{x}, t_n, \gamma(\eta\mathbf{x}, t_n)) = v_i(\mathbf{x}, t_n, \gamma(\mathbf{x}, t_n))$. Specifically,

$$V_i(\eta\mathbf{x}, t_n) = v_i(\eta\mathbf{x}, t_n, \gamma(\eta\mathbf{x}, t_n)) = \eta v_i(\mathbf{x}, t_n, \gamma(\eta\mathbf{x}, t_n)) = \eta v_i(\mathbf{x}, t_n, \gamma(\mathbf{x}, t_n)) = \eta V_i(\mathbf{x}, t_n).$$

□

The homogeneity of the pricing problem allows us to reduce it from a system of coupled three-dimensional PDEs to a system of coupled two-dimensional PDEs. By Theorem 4.3, for $\eta > 0$,

$$V_i(S, W, D, t) = \frac{1}{\eta} V_i(\eta S, \eta W, \eta D, t).$$

Suppose $W > 0$. Choosing $\eta = W^\star / W$ with $W^\star > 0$ yields

$$V_i(S, W, D, t) = \frac{W}{W^\star} V_i\left(\frac{W^\star}{W} S, W^\star, \frac{W^\star}{W} D, t\right), \tag{4.1}$$

which reveals that we need only solve the problem for two values of the withdrawal benefit: $W^\star$ and zero. We refer to this reduction in dimensionality as a *similarity reduction*.

Theorem 4.7 (Utility homogeneity under consumption-optimal strategy)**.** *Suppose that a consumption-optimal strategy is employed by the policyholder, and that for all regimes $i \in \mathcal{S}$, u_i^B and u_i^C are homogeneous of order p. Then, $\mathbf{V}(\mathbf{x}, t)$ and $\overline{\mathbf{V}}(\mathbf{x}, t)$ are homogeneous of orders 1 and p, respectively, in $\mathbf{x}$.*

The proof of this is almost identical to that of Theorem 4.3, and is hence left to the interested reader. It should be noted that the above assumes that ties in strategies are broken as in Remark 3.1.

Corollary 4.8 (Power law homogeneity)**.** *For all regimes $i \in \mathcal{S}$, take $b_i = 0$ and $p_i = p$ in (3.10) for some constant $p \neq 0$. Suppose that a consumption-optimal strategy is employed by the policyholder. Then, $\mathbf{V}(\mathbf{x}, t)$ and $\overline{\mathbf{V}}(\mathbf{x}, t)$ are homogeneous of order 1 and p, respectively, in $\mathbf{x}$.*

Proof. This follows directly from Theorem 4.7 and the fact that $u_i^C(x; a, b, p)$ is homogeneous of order p in x and b. □

This encompasses a large family of economically relevant functions, namely the power law (a.k.a. isoelastic) utility functions. Under power law utility, we can reduce the system of three-dimensional PDEs to a system of two-dimensional PDEs. As before, we get

$$\overline{V}_i(S, W, D, t) = \left(\frac{W}{W^\star}\right)^p \overline{V}_i\left(\frac{W^\star}{W} S, W^\star, \frac{W^\star}{W} D, t\right),$$

along with (4.1) whenever $W > 0$ and $W^\star > 0$.

4.2 Localized problem and boundary conditions

We approximate the original problem, posed on $(S, W, D, t) \in \mathbb{R}^3_{\geqslant 0} \times [0, T]$, on the truncated domain

$$(S, W, D, t) \in [0, S_{\text{Max}}] \times \mathcal{W} \times [0, D_{\text{Max}}] \times [0, T],$$

where $\mathcal{W} = [0, \infty)$ when a similarity reduction is applied and $\mathcal{W} = [0, W_{\text{Max}}]$ otherwise. We clamp regime-switching jumps that drive the underlying above S_{Max}. That is, we take $\min(J_{i \to j} S, S_{\text{Max}})$ (instead of $J_{i \to j} S$) to be the value of the investment account after a jump from regime i to j. No boundary conditions are needed at $S = 0$, $W = 0$, $D = 0$, $W = W_{\text{Max}}$ and $D = D_{\text{Max}}$. That is, it is sufficient to substitute one of the aforementioned boundary values of S, W or D into (2.11) and (3.6) to retrieve the relevant behaviour. At $S = S_{\text{Max}}$, for each W and D, we impose instead the linearity conditions (Windcliff et al. 2004)

$$V_i(S_{\text{Max}}, W, D, t) = C_i(t) S_{\text{Max}} \text{ and } \overline{V}_i(S_{\text{Max}}, W, D, t) = \overline{C}_i(t) S_{\text{Max}} \; \forall i \in \mathcal{S}$$

in an attempt to estimate the true asymptotic behaviour of the contract. Substituting the above into (2.11) and (3.6) yields two ordinary differential equations (ODEs) in which C_i and $\overline{C}_i$ are the dependent variables. These are solved numerically alongside the rest of the domain. Errors introduced by the above approximations are small in the region of interest, as verified by numerical experiments. At $t = T$, (2.1) and (3.2) suggest

$$V_i(S, W, D, T) = \overline{V}_i(S, W, D, T) = 0 \; \forall i \in \mathcal{S}.$$

We use Crank-Nicolson time-stepping with Rannacher smoothing (Rannacher 1984). We discretize the diffusive term using a second-order centered difference, while the convective term is discretized using a centered difference only when the corresponding backward Euler scheme is monotone. Otherwise, an upwind discretization is employed. Variable-size timestepping is used (see Johnson (2009) for an expository treatment). The resulting linear system is solved using fixed-point iteration. The details of this approach are described by d'Halluin et al. (2005) and Kennedy (2007).

4.3 Determining the hedging cost fee

At contract inception, the withdrawal and death benefits are set to the initial value of the investment account, $S(0)$. That is, $W(0) = S(0)$ and $D(0) = S(0)$. If we overload our previous definition of V_i as parameterized by the fee, α_R, the problem becomes one of determining α_R such that

$$V_I\left(S(0), W(0), D(0), 0; \alpha_R\right) - \underbrace{\mathcal{R}(0)}_{1} S(0) = 0, \tag{4.2}$$

where I is the regime observed at time zero. This is a requirement stating that α_R must be selected so as to compensate the writer for the hedging costs. We term such a value of α_R the *hedging cost fee*. Equation (4.2) is solved numerically using Newton's method.

5 Results

We begin by performing experiments under the assumptions (i) that the policyholder behaves so as to maximize the writer's losses and (ii) that the policyholder always withdraws at the contract rate. We consider a handful of numerical tests based on perturbations to the base case data in Table 5.1. We subsequently move to considering consumption-optimal strategies, in which we use the base case data in Tables 5.1 and 5.3. Throughout this section, various rates are presented in basis points (bps).

5.1 Loss-maximizing and contract rate withdrawal

All tests in this section are performed on perturbations to the base case data in Table 5.1. Table 5.2 documents wide variation in the hedging cost fee across different volatility and interest rate parameters for the two regimes considered, and for the cases with a ratcheting death benefit, with a nonratcheting death benefit, and without a death benefit. Of course, in any otherwise identical scenario, the loss-maximizing withdrawal assumption results in a higher fee since this represents the worst case scenario for the insurer. As we might expect, higher volatility is associated with an increase in the cost of hedging and thus a higher fee. The fee is also quite sensitive to the levels of the risk-free interest rate across the two regimes. The presence of a death benefit results in a notably increased fee, particularly if this feature is ratcheting.

Withdrawal analysis. We now turn to a brief exploration of loss-maximizing withdrawal strategies by the policyholder. Figures 5.1 and 5.2 show these strategies under each regime (Table 5.1) at $t = 1, 2, \ldots, 6$ assuming that the corresponding hedging cost fee is charged for hedging the contract and that $D = 100$. In either regime, if W is much bigger than S, the strategy always involves

Parameter			Value	
Volatility	σ_1	σ_2	0.0832	0.2141
Risk-free rate	r_1	r_2	0.0521	0.0521
Rate of transition	$q^{\mathbb{Q}}_{1\to2}$	$q^{\mathbb{Q}}_{2\to1}$	0.0525	0.1364
Jump magnitude	$J_{1\to2}$	$J_{2\to1}$	1	1
Initial regime	I			1
Initial investment	$S(0)$			100
Management rate	α_M			100 bps
Contract rate	G			0.05
Bonus rate	B			0.05
Initial age	x_0			65
Expiry time	T			57
Mortality data				Pasdika et al. (2005)
Ratchets				Triennial
Withdrawals				Annual

Time t	Penalty $\kappa(t)$
1	0.03
2	0.02
3	0.01
$\geqslant 4$	0

TABLE 5.1: *Pricing system base case data with regime-dependent parameters obtained from O'Sullivan and Moloney (2010) by calibration to FTSE 100 options in January 2007.*

	Hedging cost fee α_R (bps)					
Parameters	Ratcheting Death Benefit		Nonratcheting Death Benefit		No Death Benefit	
Base case (Table 5.1)	54	48	37	24	27	19
Initial regime = 2	158	113	139	75	86	52
$(r_1, r_2) = (0.04, 0.06)$	79	72	62	43	44	33
$(r_1, r_2) = (0.03, 0.07)$	124	114	106	76	73	57
$(r_1, r_2) = (0.02, 0.08)$	239	212	224	156	129	104
$(\sigma_1, \sigma_2) = (0.10, 0.20)$	62	56	45	29	31	22
$(\sigma_1, \sigma_2) = (0.15, 0.25)$	133	123	107	69	70	51

TABLE 5.2: *The value of the hedging cost fee for perturbations to the data in Table 5.1. For each perturbation, fees are calculated under the loss-maximizing (left) and contract rate withdrawal (right) strategies. Values are reported to the nearest basis point.*

withdrawing at the contract rate, but the strategy in other regions can be quite complex. We note that in the less volatile regime (Figure 5.1), the withdrawal strategy does not involve surrender for $t \leqslant 3$, prior to the vanishing of surrender charges at $t > 3$ (Table 5.1). However, in the more volatile regime (Figure 5.2), the policyholder is more willing to surrender the contract, despite the large penalties at times $t = 1$ and $t = 2$. Also note that in this regime, the policyholder's willingness to surrender (for large values of S) vanishes at $t = 3$ in anticipation of the triennial ratchet. The complexity of these loss-maximizing strategies provides some further motivation for our consumption-based approach, since it may seem implausible that individual policyholders would actually implement such strategies.

Management rate. Figure 5.3 shows the relationship between the hedging cost fee and the management rate (i.e. the proportional management expense fee α_M). As is to be expected, the fee grows superlinearly as a function of the management rate, since the management rate acts as a drag on the investment account. This confirms the observation by Forsyth and Vetzal (2013) that the use of mutual funds with high management fees as the underlying investment for variable annuities results in higher costs for the insurer compared to a policy written on funds with low management fees (e.g. exchange-traded index funds). We also see that for both the loss-maximizing and contract rate withdrawal strategy, the death benefit adds significant value to the contract, consistent with the results reported in Table 5.2. Again, the disparity between the ratcheting and nonratcheting death benefit is even more pronounced.

Alternate fee structure. Some insurers have adopted alternate fee structures that are functions of the auxiliary accounts. In general, the risky account evolves according to

$$dS = \left(\mu S - \alpha F\left(S, W, D, t\right)\right) dt + \sigma S dZ.$$

A comparison of the usual fee structure $F = S$ with $F = S \vee W$ on a contract without death benefits for various values of the management rate α_M under the loss-maximizing strategy is shown in Figure 5.4. We see that for sufficiently small management rates, the alternate fee structure reduces the hedging cost fee. However, as the management fee increases, the fee calculated under the alternate fee structure surpasses its vanilla counterpart. When the management rate is relatively low, it has a comparatively small impact in terms of decreasing the value of the investment account and hence exerts limited influence on the value of the guarantee. Moreover, since the total rate $\alpha = \alpha_M + \alpha_R$ applies to the greater of the investment account and the guarantee benefit, the size of the fee in such cases is comparatively small. However, as the management rate increases, the value of the guarantee rises and eventually a higher fee is needed to fund the cost of hedging.

5.2 Consumption-optimal withdrawal

All tests in this section are performed on perturbations to the base case data in Tables 5.1 and 5.3.

Risk-aversion. Suppose the management rate, α_M, is zero. If for all regimes $i \in \mathcal{S}$ we take the parameterization shown in Table 5.4, the consumption-optimal strategy reduces to the loss-maximizing strategy (this can be verified by direct substitution). Reflecting this, we refer to this parameterization as the *degeneracy parameterization*. Since the degeneracy parameterization corresponds to the loss-maximizing strategy, it is guaranteed to yield the highest possible hedging

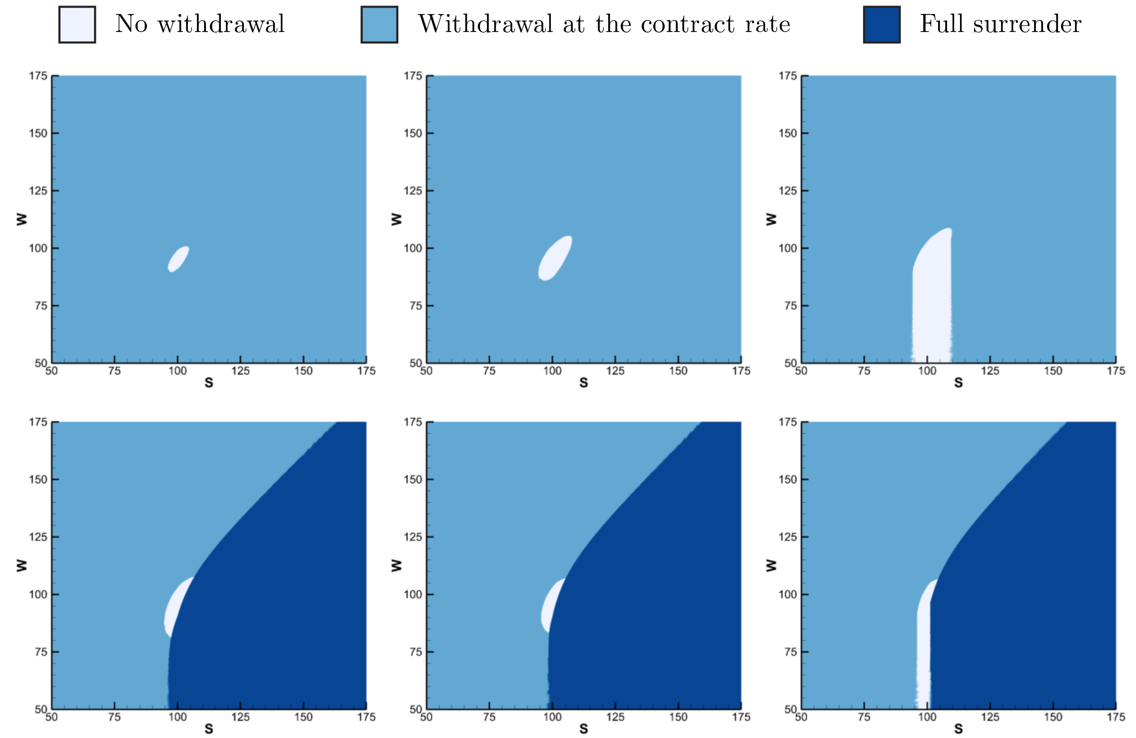


FIGURE 5.1: *Observed loss-maximizing strategies at $D = 100$ under regime 1. The hedging cost fee $\alpha_R \approx 37$ bps is used (Table 5.2). The subfigures, from top-left to bottom-right, correspond to $t = 1, 2, \ldots, 6$.*

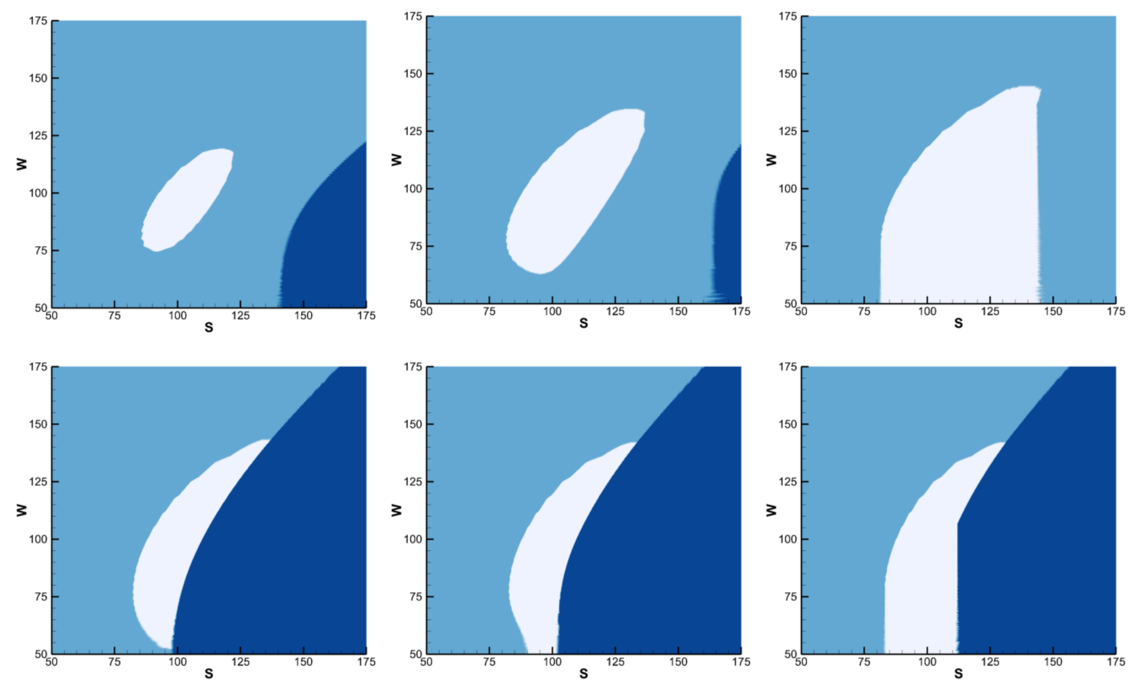
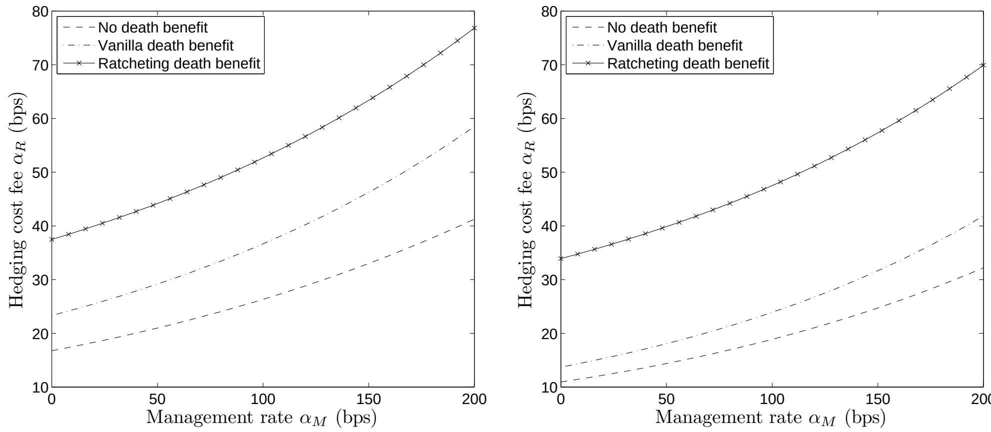


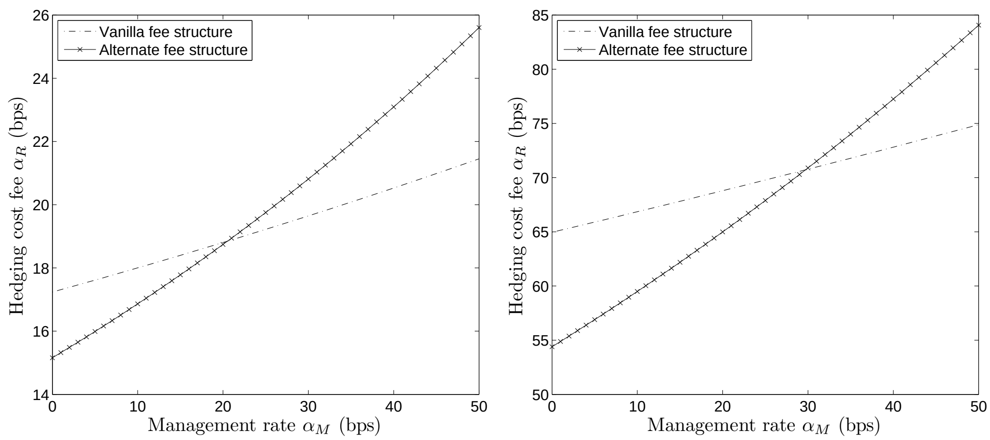
FIGURE 5.2: *Observed loss-maximizing strategies at $D = 100$ under regime 2. The hedging cost fee $\alpha_R \approx 139$ bps is used (Table 5.2). The subfigures, from top-left to bottom-right, correspond to $t = 1, 2, \ldots, 6$.*



(a) *Loss-maximizing strategy*

(b) *Contract-rate strategy*

FIGURE 5.3: *Sensitivity of hedging cost fee to the management rate.*



(a) *Initial regime $I = 1$; no death benefit*

(b) *Initial regime $I = 2$; no death benefit*

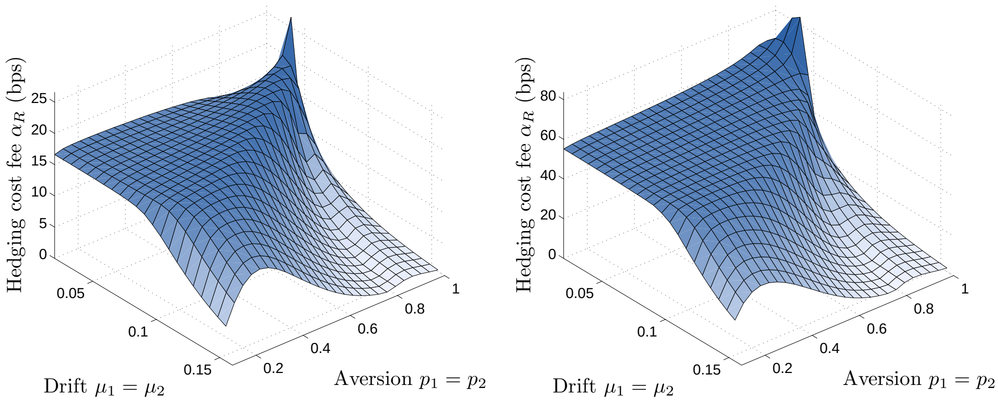
FIGURE 5.4: *Sensitivity of hedging cost fee to the management rate for different fee structures.*

Parameter			Value	
Drift rate	μ_1	μ_2	0.1	0.1
Time preference	β_1	β_2	0.032	0.032
HARA scaling	a_1	a_2	1	1
HARA offset	b_1	b_2	0	0
Risk-aversion	p_1	p_2	0.5	0.5
Bequest motive	h_1	h_2	1	1
Rate of transition	$q_{1\to2}$	$q_{2\to1}$	0.0525	0.1364

TABLE 5.3: *Consumption system base case data with rate of time preference obtained from Nishiyama and Smetters (2005).*

Parameter	α_M	μ_i	β_i	a_i	b_i	p_i	h_i
Value	0	$r_i - \rho_i^{\mathbb{Q}}$	r_i	1	0	1	1

TABLE 5.4: *Degeneracy parameterization.*



(a) *Initial regime $I = 1$; no death benefit*

(b) *Initial regime $I = 2$; no death benefit*

FIGURE 5.5: *Effects of varying drift and risk-aversion on the hedging cost fee.*

cost fee. We stress that this holds only when the management rate is zero, as in Table 5.4. The utility parameters under this parameterization $u_i^B(x) = h_i u_i^C(x; a_i = 1, b_i = 0, p_i = 1)$ correspond to the case of risk-neutral utility: $u_i^B(x) = u_i^C(x) = x$.

Although the above only holds under the degeneracy parameterization, we expect to see large hedging cost fees under parameterizations that are close to the degeneracy parameterization. Figure 5.5 shows the effect of simultaneously varying the regime-dependent drifts μ_1 and μ_2 and risk-aversion parameters p_1 and p_2 on the hedging cost fee for the base case data in Tables 5.1 and 5.3 for a contract without death benefits. When $\mu_1 = \mu_2 = 0.0521$ and $p_1 = p_2 = 1$, a global maximum appears on each surface. As expected, the parameterization $\mu_1 = \mu_2 = 0.0521$ and $p_1 = p_2 = 1$ is close to the degeneracy parameterization (Tables 5.1 and 5.3 specify $\alpha = 100$ bps ≈ 0 and $\beta_i = 0.032 \approx 0.0521 = r_i$), and hence these maxima (27 bps and 84 bps, rounded to the nearest basis point) are very close to the hedging cost fees for each regime calculated under the loss-maximizing strategy (27 and 86 bps, rounded to the nearest basis point; see Table 5.2). Realistically, these maxima are not of great interest to the insurer as they occur where the drift of the investment account is equal to the risk-free rate of return. More interestingly, both surfaces exhibit a large "plateau" region (i.e. where the gradient is approximately zero) for which the consumption-optimal hedging cost fee is close to that calculated under the contract rate withdrawal strategy. This suggests that for a large family of parameters, the policyholder withdraws at nearly the contract rate. This can be verified by comparing the hedging cost fee here for the two regimes with those shown in Table 5.2 (19 bps and 52 bps, rounded to the nearest basis point).

Taxation. It has been suggested by Moenig and Bauer (2011) that a policyholder's strategy depends on the taxation of their withdrawals. We assume that withdrawals are taxed on the American *last-in first-out* (LIFO) basis and that earnings in the underlying investment account

	0%	10%	20%	30%	40%	50%
Initial regime $I = 1$	18.0	18.9	19.2	18.7	17.7	16.3
Initial regime $I = 2$	54.7	55.8	56.3	56.7	57.0	57.2

TABLE 5.5: *Sensitivity of the hedging cost fee to the tax rate. Values are reported to the nearest tenth of a basis point.*

grow on a tax-deferred basis.

This requires the addition of another process $Q(t)$, which is referred to as the *tax base* at time t. The tax base denotes what amount of the underlying investment account is nontaxable. Initially, $Q(0) = S(0)$. Q is piecewise constant between withdrawals. When a withdrawal of size w is made at time t,

$$Q\left(t^{+}\right) = Q\left(t^{-}\right) - \underbrace{\left(w - \left[S\left(t^{-}\right) - Q\left(t^{-}\right)\right] \vee 0\right) \vee 0}_{\text{Nontaxable portion of the withdrawal}}.$$

The introduction of the tax base variable introduces an additional dimension for which the PDEs must be solved. We assume that policyholders optimize their after-tax consumption. Table 5.5 shows the effect of the tax rate on the hedging cost fee for the base case contract without death benefits. We find that for typical levels of risk-aversion, taxation has a small effect on the fee. Even for extreme tax rates of 50%, the fee changes by at most several basis points.

6 Conclusion

We have introduced a general methodology that allows for the decoupling of policyholder behaviour from the pricing (i.e. determining the cost of hedging) of a variable annuity. Assuming that the underlying investment follows a regime-switching process, this yields two weakly coupled systems of PDEs: the pricing and utility systems. When considering strategies contingent on the policyholder's level of consumption, the utility system is used to generate policyholder withdrawal behaviour, which is in turn fed into the pricing system as a means to determine the cost of hedging the contract. Our methodology is general enough to allow us to consider any withdrawal strategy contingent on either the cost of hedging the contract or the policyholder's level of consumption.

We have adopted the GLWDB as a case study. A similarity reduction transforms our systems of three-dimensional PDEs to systems of two-dimensional PDEs, allowing us to generate numerical solutions with speed. In the absence of a death benefit, these systems further simplify into systems involving one-dimensional PDEs, which (for a reasonable number of regimes) can be solved with minimal computational effort.

Since GLWDB contracts are held over long periods of time, regime-switching serves as a natural model for the process followed by the underlying asset. This process can incorporate stochastic interest rates and volatility in a simple and intuitive manner. It is also possible to have policyholder preferences which differ between regimes. Results obtained under various regime-switching processes indicate that the hedging cost fee is extremely sensitive to the regime-dependent parameters.

We show that the inclusion of a death-benefit yields large fees for typical contract values under both the loss-maximizing strategy and the static strategy of always withdrawing at the contract rate. We observe an even more pronounced disparity between the no-arbitrage fee generated by a contract with nonratcheting death benefits compared to a contract with ratcheting death benefits. These findings are consistent with the phasing out of products including ratcheting death benefits from the Canadian market.

We find that for a large family of utility functions, the consumption-optimal strategy yields a hedging cost fee that is very close to the hedging cost fee calculated by assuming that the policyholder withdraws at the contract rate. This can be understood as substantiating the otherwise seemingly naïve assumption that the policyholder "generally" withdraws at the contract rate. Adopting the contract rate withdrawal strategy renders the pricing problem computationally simple, as this strategy is deterministic and can easily be implemented in either the PDE or an equivalent Monte Carlo formulation.

Appendix

In this Appendix, we derive the no-arbitrage regime-switching PDEs for general contingent claims. Following along the lines of this Appendix, the reader should have no difficulty combining these arguments with those in Section 2 to obtain the final equation (2.11).

A Regime-switching model

A.1 Regime-switching PDEs

Consider the M-regime process S evolving according to

$$dS\left(t\right)=a_i\left(S\left(t\right),t\right)dt+b_i\left(S\left(t\right),t\right)dZ\left(t\right)+\sum_{j=1}^{M}S\left(t\right)\left(J_{i\to j}-1\right)dX_{i\to j}\left(t\right)$$

in which dS describes the increment of S assuming that the regime at time t is i. We restrict $J_{i\to i}=1$ for all i so that jumps in the underlying are not experienced unless there is a change in regime.

In the relevant literature, it is often mentioned that the introduction of the regime-switching underlying S yields an incomplete market (Zhou and Yin 2003, Elliott et al. 2005), if the hedging portfolio contains only the underlying asset and the risk-free account. We consider instead a complete market consisting of the bond and M independent hedging instruments. Note that the assumption of the availability of M instruments is not farfetched; we need only find M instruments written on the regime-switching underlying S. Often, it is possible to take S itself as one of these instruments (this scenario is detailed in §A.2).

We follow the formulation of a regime-switching framework as derived by Kennedy (2007). Consider a portfolio Π short an option V and with positions in instruments $F^{(1)}$, $F^{(2)}$, …, $F^{(M)}$. We assume that the trading instruments depend only on $S\left(t\right)$ and t. Let B represent the money market process with risk-free rate r (i.e. $dB=rBdt$). Denote by V_i and $F_i^{(k)}$ the values of the option and k^{th} instrument in regime i. Assuming that regime i is observed at time t,

$$\Pi\left(S\left(t\right),t\right)=-V_i\left(S\left(t\right),t\right)+\sum_{k=1}^{M}\left[\omega^{(k)}F_i^{(k)}\left(S\left(t\right),t\right)\right]+B\left(t\right). \tag{A.1}$$

The increment of the above portfolio can be written as

$$d\Pi\left(S\left(t\right),t\right) = -dV_i\left(S\left(t\right),t\right) + \sum_{k=1}^{M}\left[\omega^{(k)} dF_i^{(k)}\left(S\left(t\right),t\right)\right] + dB\left(t\right). \tag{A.2}$$

where

$$\begin{aligned}
dV_i &= \hat{\mu}_i dt + \hat{\sigma}_i dZ + \sum_{j=1}^{M} \Delta V_{i\to j} dX_{i\to j} \\
\hat{\mu}_i &= \frac{1}{2} b_i^2 \frac{\partial^2 V_i}{\partial S^2} + a_i \frac{\partial V_i}{\partial S} + \frac{\partial V_i}{\partial t} \\
\hat{\sigma}_i &= b_i \frac{\partial V_i}{\partial S} \\
\Delta V_{i\to j} &= V_j\left(J_{i\to j} S, t\right) - V_i\left(S, t\right)
\end{aligned}$$

and

$$\begin{aligned}
dF_i^{(k)} &= \bar{\mu}_i^{(k)} dt + \bar{\sigma}_i^{(k)} dZ + \sum_{j=1}^{M} \Delta F_{i\to j}^{(k)} dX_{i\to j} \\
\bar{\mu}_i^{(k)} &= \frac{1}{2} b_i^2 \frac{\partial^2 F_i^{(k)}}{\partial S^2} + a_i \frac{\partial F_i^{(k)}}{\partial S} + \frac{\partial F_i^{(k)}}{\partial t} \\
\bar{\sigma}_i^{(k)} &= b_i \frac{\partial F_i^{(k)}}{\partial S} \\
\Delta F_{i\to j}^{(k)} &= F_j^{(k)}\left(J_{i\to j} S, t\right) - F_i^{(k)}\left(S, t\right).
\end{aligned}$$

Substituting these expressions into (A.2) yields

$$\begin{aligned}
d\Pi\left(t\right) = &\left[\sum_{k=1}^{M}\left[\omega^{(k)} \bar{\mu}_i^{(k)}\right] + rB - \hat{\mu}_i\right] dt + \left[\sum_{k=1}^{M}\left[\omega^{(k)} \bar{\sigma}_i^{(k)}\right] - \hat{\sigma}_i\right] dZ \\
&+ \sum_{j=1}^{M}\left[\sum_{k=1}^{M}\left[\omega^{(k)} \Delta F_{i\to j}^{(k)}\right] - \Delta V_{i\to j}\right] dX_{i\to j}.
\end{aligned} \tag{A.3}$$

To make the portfolio deterministic, we eliminate Brownian risk by

$$\sum_{k=1}^{M} \omega^{(k)} \bar{\sigma}_i^{(k)} = \hat{\sigma}_i \tag{A.4}$$

and jump risk by

$$\sum_{k=1}^{M} \omega^{(k)} \Delta F_{i\to j}^{(k)} = \Delta V_{i\to j} \ \forall j \in \mathcal{S}. \tag{A.5}$$

Note that the jump risk equation corresponding to $j = i$ relates a zero change in the hedging instruments to zero change in the option, so that to eliminate jump risk, we need only satisfy $M - 1$ equations.

Given that the portfolio is deterministic, the principle of no-arbitrage requires $r\Pi dt = d\Pi$. Using the expressions (A.1) and (A.3), we write this as

$$\sum_{k=1}^{M} \omega^{(k)} \left(\bar{\mu}_i^{(k)} - rF_i^{(k)} \right) = \hat{\mu}_i - rV_i. \tag{A.6}$$

Equations (A.4), (A.5) and (A.6) make for a total of $M+1$ equations in M unknowns. This system has a solution if and only if one of the equations is a linear combination of the others. We denote by ξ_i, $q^{\mathbb{Q}}_{i\to 1}$, $q^{\mathbb{Q}}_{i\to 2}$, …, $q^{\mathbb{Q}}_{i\to M}$ the weights under which the linear dependence requirement

$$\xi_i \left(\sum_{k=1}^{M} \left[\omega^{(k)} \bar{\sigma}_i^{(k)} \right] - \hat{\sigma}_i \right) = \sum_{\substack{j=1 \\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j} \left(\sum_{k=1}^{M} \left[\omega^{(k)} \Delta F^{(k)}_{i\to j} \right] - dV_{i\to j} \right) + \sum_{k=1}^{M} \left[\omega^{(k)} \left(\bar{\mu}^{(k)} - rF_i^{(k)} \right) \right] - (\hat{\mu}_i - rV_i)$$

holds true. Rearranging this expression,

$$\sum_{k=1}^{M} \left[\omega^{(k)} \left(\xi_i \bar{\sigma}_i^{(k)} - \sum_{\substack{j=1 \\ j\neq i}}^{M} \left[q^{\mathbb{Q}}_{i,j} \Delta F^{(k)}_{i\to j} \right] - \left(\bar{\mu}_i - rF_i^{(k)} \right) \right) \right] - \xi_i \hat{\sigma}_i + \sum_{\substack{j=1 \\ j\neq i}}^{M} \left[q^{\mathbb{Q}}_{i\to j} \Delta V_{i\to j} \right] + \hat{\mu}_i - rV_i = 0.$$

Since this must hold for any position $\omega^{(1)}$, $\omega^{(2)}$, …, $\omega^{(M)}$, we write the above as

$$\xi_i \bar{\sigma}_i^{(k)} - \sum_{\substack{j=1 \\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j} \Delta F^{(k)}_{i\to j} = \left(\bar{\mu}_i^{(k)} - rF_i^{(k)} \right) \ \forall k \in \mathcal{S} \tag{A.7}$$

and

$$\xi_i \hat{\sigma}_i - \sum_{\substack{j=1 \\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j} \Delta V_{i\to j} = \hat{\mu}_i - rV_i. \tag{A.8}$$

This procedure effectively decouples the hedging instruments from the option V. Resolving the symbols $\hat{\mu}_i$ and $\hat{\sigma}_i$ in (A.8) yields

$$\frac{1}{2} b_i^2 \frac{\partial^2 V_i}{\partial S^2} + (a_i - \xi_i b_i) \frac{\partial V_i}{\partial S} - rV_i + \sum_{\substack{j=1 \\ j\neq i}}^{M} \left[q^{\mathbb{Q}}_{i\to j} \Delta V_{i\to j} \right] + \frac{\partial V_i}{\partial t} = 0, \tag{A.9}$$

which describes a system of M PDEs: one for each regime. The more familiar form above reveals $a_i - \xi_i b_i$ as the risk-neutral drift and the $q^{\mathbb{Q}}_{i\to j}$ terms as the risk-neutral transition intensities.

We express this more compactly by defining

$$q^{\mathbb{Q}}_{i\to i} = -\sum_{\substack{j=1\\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j}$$

and noting that

$$\sum_{\substack{j=1\\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j}\Delta V_{i\to j} = \sum_{\substack{j=1\\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j} V_j\left(J_{i\to j}S, t\right) - V_i \sum_{\substack{j=1\\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j} = \sum_{\substack{j=1\\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j} V_j\left(J_{i\to j}S, t\right) + q^{\mathbb{Q}}_{i\to i} V_i$$

so that (A.9) becomes

$$\frac{1}{2} b_i^2 \frac{\partial^2 V_i}{\partial S^2} + \left(a_i - \xi_i b_i\right)\frac{\partial V_i}{\partial S} - \left(r - q^{\mathbb{Q}}_{i\to i}\right) V_i + \sum_{\substack{j=1\\ j\neq i}}^{M}\left[q^{\mathbb{Q}}_{i\to j} V_j\left(J_{i\to j}S, t\right)\right] + \frac{\partial V_i}{\partial t} = 0. \tag{A.10}$$

A.2 Eliminating the market price of risk

It is often possible to eliminate the market price of risk $\xi_i b_i$ from (A.10) (Kennedy 2007). For example, let

$$a_i\left(S\left(t\right), t\right) = \left(\mu_i - \alpha\right) S\left(t\right)$$

and

$$b_i\left(S\left(t\right), t\right) = \sigma_i S\left(t\right).$$

Under these parameters, (A.10) becomes

$$\frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + \left(\mu_i - \alpha - \xi_i\sigma_i\right) S\frac{\partial V_i}{\partial S} - \left(r - q^{\mathbb{Q}}_{i\to i}\right) V_i + \sum_{\substack{j=1\\ j\neq i}}^{M}\left[q^{\mathbb{Q}}_{i\to j} V_j\left(J_{i\to j}S, t\right)\right] + \frac{\partial V_i}{\partial t} = 0. \tag{A.11}$$

Suppose further that S itself is not tradeable but tracks the tradeable index $\hat{S}$ with

$$d\hat{S}\left(t\right) = \mu_i \hat{S}\left(t\right) dt + \sigma_i \hat{S}\left(t\right) dZ\left(t\right).$$

Take the 1st instrument, $F^{(1)}$, to be $\hat{S}$ so that

$$\begin{aligned}
\bar{\mu}_i^{(1)} &= \mu_i \hat{S}\\
\bar{\sigma}_i^{(1)} &= \sigma_i \hat{S}\\
\Delta F^{(1)}_{i\to j} &= \hat{S}\left(J_{i\to j} - 1\right).
\end{aligned}$$

Substituting this into (A.7) for $k = 1$ yields

$$\xi_i\sigma_i\hat{S} - \sum_{\substack{j=1\\ j\neq i}}^{M} q^{\mathbb{Q}}_{i\to j}\hat{S}\left(J_{i\to j} - 1\right) = \xi_i\sigma_i\hat{S} - \rho_i^{\mathbb{Q}}\hat{S} = \mu_i\hat{S} - r\hat{S}.$$

More compactly, we write this as

$$\xi_i \sigma_i \hat{S} = \left(\rho_i^{\mathbb{Q}} + \mu_i - r \right) \hat{S} \tag{A.12}$$

where

$$\rho_i^{\mathbb{Q}} = \sum_{\substack{j=1 \\ j \neq i}}^{M} q_{i \to j}^{\mathbb{Q}} \left(J_{i \to j} - 1 \right) = \sum_{j=1}^{M} q_{i \to j}^{\mathbb{Q}} J_{i \to j}.$$

Whenever $\hat{S}$ is equal to zero, S is necessarily zero so that the term associated with the market price of risk in (A.11) also vanishes. We are thus only interested in the case in which $\hat{S} \neq 0$, under which (A.12) states that

$$\xi_i \sigma_i = \rho_i^{\mathbb{Q}} + \mu_i - r.$$

Substituting the above into (A.11),

$$\frac{1}{2} \sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + \left(r - \alpha - \rho_i^{\mathbb{Q}} \right) S \frac{\partial V_i}{\partial S} - \left(r - q_{i \to i}^{\mathbb{Q}} \right) V_i + \sum_{\substack{j=1 \\ j \neq i}}^{M} \left[q_{i \to j}^{\mathbb{Q}} V_j \left(J_{i \to j} S, t \right) \right] + \frac{\partial V_i}{\partial t} = 0.$$

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