Numerical implicitization of parametric hypersurfaces with linear algebra *

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Abstract. We present a new method for implicitization of parametric curves, surfaces and hypersurfaces using essentially numerical linear algebra. The method is applicable for polynomial, rational as well as trigonometric parametric representations. The method can also handle monoparametric families of parametric curves, surfaces and hypersurfaces with a small additional amount of human interaction. We illustrate the method with a number of examples. The efficiency of the method compares well with the other available methods for implicitization.

1 Introduction

The problem of implicitization for curves, surfaces and hypersurfaces is an important problem in Algebraic Geometry with immediate practical applications in such areas as Geometric Modeling, Graphics, Computer Aided Geometric Design (see [Hof89]). The implicitization problem has been addressed using a variety of mathematical methodologies and techniques including Gröbner bases, (see [Buc88], [Kal90], [GC92], [LM94], [FHL96]) Characteristic sets, (see [Li89], [Gao00]) Resultants, (see [SAG84], [CG92]) Perturbation, (see [Hob91], [MC92a], [MC92b], [SSQK94], [Hon97], [SGD97]), Multidimensional Newton formulae (see [GV97]), Elimination theories, (see [SAG85], [Wan95]) and Symmetric functions (see [GVT95]).

We note that the inverse problem of parameterization, which is an equally important problem in Algebraic Geometry with direct practical applications, has also been investigated by many authors (see for example [AB88], [AGR95], [HS98], [Sch98], [SW91], [SW98]).

Some of the above methods work for special categories of curves, surfaces and hypersurfaces. Moreover, some methods handle only special kinds of parametric representations, like polynomial, rational or trigonometric ones.

It is important to have efficient algorithms to solve the implicitization and parameterization problems. This is mainly because in many practical applications

* Work supported by the Ontario Research Centre for Computer Algebra the Ontario Research and Development Challenge Fund and the Natural Sciences and Engineering Research Council of Canada.

and depending on the particular circumstances we want to use the parametric equations or the implicit equation.

In this paper we present a new implicitization method for curves, surfaces and hypersurfaces that works for polynomial, rational and trigonometric parameterizations. The method uses an alternative interpretation of the implicitization problem as an eigenvalue problem, inspired by theoretical considerations coming from the area of the Calculus of Variations (see [Tro83]). The method ultimately uses numerical linear algebra to recover the implicit (cartesian) equation from the parametric equations.

2 Description of the problem

In what follows the term, geometric object, will be used to describe a curve, a surface, or a general hypersurface.

A parameterization of a geometric object in a space of dimension n can be described by the following set of parametric equations:

$$x_1 = f_1(t_1, \ldots, t_k), \ldots, x_n = f_n(t_1, \ldots, t_k) \tag{1}$$

where the $t_1, \ldots, t_k$ are parameters and the functions $f_1, \ldots, f_n$ can be polynomial, rational or trigonometric functions. The case $n = 2$ corresponds to curves, the case $n = 3$ corresponds to surfaces and the case $n \geq 4$ corresponds to hypersurfaces in general. The implicitization problem consists in computing the polynomial cartesian (implicit) equation

$$p(x_1, \ldots, x_n) = 0, \tag{2}$$

of the geometric object described by the parametric equations (1), which satisfies

$$p(f_1(t_1, \ldots, t_k), \ldots, f_n(t_1, \ldots, t_k)) = 0,$$

for all values of the parameters $t_1, \ldots, t_k$.

3 Implicitization as an eigenvalue problem

Suppose that $g(x, y) = M(x, y)\mathbf{a}$ where $\mathbf{a}$ is the vector of (unknown) coefficients of the polynomial $g(x, y)$ and $M(x, y) = [1, x, y, \ldots, y^m]$ where m is the total degree of $g(x, y)$. Given a parameterization $(x(s), y(s))$ of $g(x, y) = 0$, numerical or exactly-known, we can ask for the vector $\mathbf{a}$ that minimizes

$$\begin{aligned} J(g) &= \int_{s_0}^{s_1} w(s) g^* g \, ds \\ &= \int_{s_0}^{s_1} w(s)\mathbf{a}^* M^*(x(s), y(s)) M(x(s), y(s))\mathbf{a} \, ds \end{aligned}$$

(for a specified positive weight function $w(s)$) subject to the constraint $\|\mathbf{a}\|^2 = 1$. Forming the Lagrange multiplier we get the standard Rayleigh-Ritz problem of minimizing $K(\mathbf{a}) = J + \lambda(1 - \mathbf{a}^*\mathbf{a})$.

Consider $K(\mathbf{a}+\Delta\mathbf{a}) - K(\mathbf{a}) =$

$$2\Delta\mathbf{a}^*\,(G-\lambda I)\,\mathbf{a} + \Delta\mathbf{a}^*\,(G-\lambda I)\,\Delta\mathbf{a} \tag{3}$$

where G is the (Hermitian, positive semi-definite) structured matrix

$$G = \int_{s_0}^{s_1} w(s) M^* M \, ds\,, \tag{4}$$

and we therefore see that if

$$[G-\lambda I]\,\mathbf{a} = 0\,, \tag{5}$$

then λ is an eigenvalue of G with eigenvector $\mathbf{a}$; therefore $\lambda \geq 0$ because G is positive semidefinite. This gives $K(\mathbf{a}+\Delta\mathbf{a}) - K(\mathbf{a}) = \Delta\mathbf{a}^*(G-\lambda I)\Delta\mathbf{a}$. Now if λ is the *smallest* eigenvalue of G, the eigenvalues of $(G-\lambda I)$ are all non-negative, and hence

$$K(\mathbf{a}+\Delta\mathbf{a}) - K(\mathbf{a}) = \Delta\mathbf{a}^*(G-\lambda I)\Delta\mathbf{a} \geq 0\,. \tag{6}$$

Thus our minimum will occur at an eigenvector of G corresponding to its smallest eigenvalue.

Moreover, the standard theory [Tro83, p. 343] shows that the *eigenvalue* λ is exactly $J(M(x,y)\mathbf{a})$ for the corresponding eigenvector $\mathbf{a}$.

Finally, equality in (6) occurs if and only if $\Delta\mathbf{a}$ is also an eigenvector corresponding to λ. This is possible only if the smallest eigenvalue is multiple.

More generally, the conditioning of these eigenvectors depends on the distances to the nearest other small eigenvalues [GVL95]. Errors are amplified by a factor of essentially $1/(\lambda_k - \lambda_m)$.

If an implicitization exists with the support $M(x,y)$, then finding a vector in the null space of G finds this implicitization.

4 Description and implementation of the method

In this section we describe the algorithmic steps of the method in pseudo-code. We have implemented the method in MAPLE and tested our implementation with all the examples given in the next section.

Input:	Parametric equations of the form (1) for specific n, k.
Output:	The cartesian (implicit) equation for the geometric object represented by these equations.
Step 1:	choose m (total degree of the implicit equation).
Step 2:	construct the line matrix v of all power products of total degree up to m in the variables $x_1, \ldots, x_n$.
Step 3:	compute the matrix $M = v^t \cdot v$.
Step 4:	substitute $x_1, \ldots, x_n$ by their parametric representations (1), in the matrix M.
Step 5:	integrate the elements of the matrix M successively over each parameter $t_1, \ldots, t_k$.
Step 6:	compute a null-vector nv of the matrix resulting from Step 5.
Step 7:	recover the implicit equation as the product $M \cdot nv$.

Several comments are in order to clarify certain points in the above description of our implicitization method.

1. The choice of the predicted total degree m of the implicit equation, can be facilitated using well-known upper bounds, such as the ones mentioned in ([GVT95], [GV97]) for example.
2. During the integration step, care should be taken so as to avoid integrals with infinite values or divergent integrals. Such degenerate cases may occur when for example the parametric equations contain denominators or trigonometric functions. Usually it is an easy matter to choose suitable intervals of integration. In the case of rational parametric equations this is the problem of base points (see for example [CG92], [MC92a], [MC92b]).
3. Another issue related to the integration step, is that sometimes it is inevitable to perform the integrations numerically, simply because the analytic expression is either too complicated to be of any use, or is not elementary. When numerical integration is employed, the resulting matrix will have floating point elements and one should be very careful about how to compute correctly the nullspace. Indeed, it may happen that according to the precision used for the computation, one obtains one or more vectors as a basis for the nullspace.
4. In the last step of the algorithm, we obtain the cartesian equation in the variables $x_1, \ldots, x_n$, but this will not always be a polynomial with integer coefficients. Some more processing is necessary to discover the integer relations among the coefficients and finally multiply by the appropriate number to unveil a polynomial with integer coefficients. This can be done using integer relation finding algorithms as they are implemented in Maple.

5 Application of the method

In this section we give some examples to illustrate the use of the implicitization method for curves and surfaces. The method works for rational as well as for trigonometric parameterizations and can also be used to recover cartesian equations for monoparametric families of curves or surfaces.
Example 1. (The Descartes Folium)
Consider the following parametric equations for the plane algebraic curve known as the Descartes Folium:

$$x = \frac{3t^2}{t^3+1}, \quad y = \frac{3t}{t^3+1}. \tag{7}$$

Choose $m = 3$ and define the line matrix $v = [1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3]$ and form the associated 10×10 matrix $M = v^t \cdot v$:

$$M = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \\ x & x^2 & xy & x^3 & x^2y & xy^2 & x^4 & x^3y & x^2y^2 & xy^3 \\ y & xy & y^2 & x^2y & xy^2 & y^3 & x^3y & x^2y^2 & xy^3 & y^4 \\ x^2 & x^3 & x^2y & x^4 & x^3y & x^2y^2 & x^5 & x^4y & x^3y^2 & x^2y^3 \\ xy & x^2y & xy^2 & x^3y & x^2y^2 & xy^3 & x^4y & x^3y^2 & x^2y^3 & xy^4 \\ y^2 & xy^2 & y^3 & x^2y^2 & xy^3 & y^4 & x^3y^2 & x^2y^3 & xy^4 & y^5 \\ x^3 & x^4 & x^3y & x^5 & x^4y & x^3y^2 & x^6 & x^5y & x^4y^2 & x^3y^3 \\ x^2y & x^3y & x^2y^2 & x^4y & x^3y^2 & x^2y^3 & x^5y & x^4y^2 & x^3y^3 & x^2y^4 \\ xy^2 & x^2y^2 & xy^3 & x^3y^2 & x^2y^3 & xy^4 & x^4y^2 & x^3y^3 & x^2y^4 & xy^5 \\ y^3 & xy^3 & y^4 & x^2y^3 & xy^4 & y^5 & x^3y^3 & x^2y^4 & xy^5 & y^6 \end{bmatrix}.$$

We substitute equations (7) into the matrix M to obtain a new matrix M'. Integrate all the elements of the matrix M' with respect to t over the interval $[0, 2]$. Since the denominators in equations (7) have a singularity at $t = -1$, we choose an interval which does not contain that point. The integrations can be performed symbolically or numerically. We prefer the numerical evaluation in this example, because the analytical expressions for the integrals yield a fairly complicated matrix. The difference in the computing times between calculating the nullvector for the analytical and the numerical matrix, is dramatic. The numerical rank of the resulting matrix is 9, which means that its nullspace is of dimension 1 and thus generated by one nullvector. The Maple environment variable $Digits$ is set to 15 in order to achieve a better accuracy. We compute the nullvector and multiply it by v, to obtain the equation

$$-0.9045\, xy + 0.3015\, x^3 + 0.3015\, y^3 = 0,$$

which shows that the implicit equation of the Descartes Folium is:

$$x^3 + y^3 - 3\ x\ y = 0.$$

The most time-consuming part of the computation (2.5 sec) is the numerical evaluation of the 100 definite integrals involved.

Example 2.

Consider the following trigonometric parametric equations of the unit sphere in three-dimensional space:

$$x = \cos\theta\ \sin\phi,\ y = \cos\theta\ \cos\phi,\ z = \sin\theta.$$

We form the 10×10 matrix with first row $[1, x, y, z, x^2, xy, xz, y^2, yz, z^2]$ and integrate from 0 to $\pi/3$ for θ and ϕ successively. The resulting matrix has rank

9 and its nullspace is spanned by the vector $[-1,0,0,0,1,0,0,1,0,1]$. This gives directly the cartesian equation of the unit sphere:

$$-1+x^2+y^2+z^2=0.$$

The method works also for rational parameterizations of the unit sphere.
Example 3.
Let a be a parameter and consider the family of curves defined by the following rational parametric equations:

$$x=\frac{t\left(a-t^2\right)}{\left(1+t^2\right)^2}\quad y=\frac{t^2\left(a-t^2\right)}{\left(1+t^2\right)^2}$$

We compute the cartesian equation for some values of a and by extrapolation we have that the general monoparametric cartesian equation for the family of curves is:

$$x^4-a\,yx^2+2\,x^2y^2+y^3+y^4=0.$$

Now an easy computation shows that this equation is indeed valid for arbitrary a.
Example 4.
Unfortunately other monoparametric families present bigger difficulties. Consider the family of curves given by the polynomial parametric equations:

$$x=t+t^2,\quad y=t+t^n \tag{8}$$

where n is a parameter. We compute the cartesian equation for some values of n and we see that we have to distinguish two cases according to the parity of n, for the general monoparametric cartesian equation of the family of the curves. For n even the cartesian equation has $n+1$ terms, and is of the form:

$$x^n+\sum_{i=2}^{k}(a_i+b_iy)x^i+y^2-nxy=0,\quad k=\frac{n}{2}, \tag{9}$$

where the a_i, b_i are constants. For n odd the cartesian equation has $n+3$ terms, and is of the form:

$$x^n+2x^{k+1}+\sum_{i=1}^{k}(c_i+d_iy)x^i-2y-y^2=0,\quad k=\frac{n-1}{2}, \tag{10}$$

where the c_i, d_i are constants. We have not solved the problem completely, like in the previous example. But now for an arbitrary positive integer n we can substitute the parametric equations (8) into (9) for even n (resp. to (10) for odd n) and determine the unknown coefficients a_i, b_i (resp c_i, d_i) by solving a highly structured linear system of $2n-1$ equations in $n-2$ (resp. $n-1$ unknowns).
Example 5.

The following example is taken from [GC92]. Consider the parametric equations for a Bézier curve:

$$x(t) = \frac{8\,t^6 - 12\,t^5 + 32\,t^3 + 24\,t^2 + 12\,t}{t^6 - 3\,t^5 + 3\,t^4 + 3\,t^2 + 3\,t + 1}$$

and

$$y(t) = \frac{24\,t^5 + 54\,t^4 - 54\,t^3 - 54\,t^2 + 30\,t}{t^6 - 3\,t^5 + 3\,t^4 + 3\,t^2 + 3\,t + 1}.$$

We form the 10×10 matrix with first row $[1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3]$ and perform the integration from $t = 1$ to $t = 2$. The resulting equation to an accuracy of 7 decimal digits is:

$$\begin{gathered} 0.0001644979\,y^3 + (0.005604679 - 0.001665542\,x)\,y^2 \\ + \left(-0.001110361\,x - 0.3527776 - 0.00003965575\,x^2\right) y \\ +0.8819439\,x + 0.02516157\,x^3 - 0.3115356\,x^2 = 0. \end{gathered}$$

If we normalize this equation by dividing with the smallest coefficient in absolute value, then we get the following equation, in which some of the integer relations between the coefficients of the final equation appear already:

$$\begin{gathered} 4.148147\,y^3 + (141.3333 - 42.00001\,x)\,y^2 \\ + \left(-1.0\,x^2 - 8896.001 - 28.0\,x\right) y \\ -7856.001\,x^2 + 22240.0\,x + 634.4999\,x^3 = 0. \end{gathered}$$

The final step is provided by either Maple (using the `convert/rational` command in conjunction with a small value of the `Digits` environment variable, say 5) or RevEng, the newer version of the Inverse Symbolic Calculator available on-line from the CECM (http://www.cecm.sfu.ca/MRG/INTERFACES.html). We disregard the integer coefficients in the above equation and after processing the remaining three non-integer coefficients we discover that:

$$4.148147 \simeq \frac{112}{27}, \quad 141.3333 \simeq \frac{424}{3} \quad 634.4999 \simeq \frac{1269}{2}$$

Multiplying the equation with the lcm of the denominators which is 54, we get the final cartesian equation:

$$\begin{gathered} 224\,y^3 + (7632 - 2268\,x)\,y^2 + \left(-1512\,x - 480384 - 54\,x^2\right) y \\ +34263\,x^3 - 424224\,x^2 + 1200960\,x = 0, \end{gathered}$$

which can easily be verified with Maple.

Example 6.

The following example is taken from [SAG85]. Consider a rational cubic Bézier curve with control points

$$P_0 = (4,1), P_1 = (5,6), P_2 = (5,0), P_3 = (6,4),$$

with respective weights $w_0 = 1, w_1 = 2, w_2 = 2, w_3 = 1$. The parametric equations of the curve are given by:

$$\begin{aligned} x &= 2\,t^3 - 18\,t^2 s + 18\,ts^2 + 4\,s^3 \\ y &= 39\,t^3 - 69\,t^2 s + 33\,ts^2 + s^3 \\ z &= -3\,t^2 s + 3\,ts^2 + s^3 \end{aligned}$$

We choose to work with a 20×20 matrix (total degree 3) and integrate from 0 to 1 for t and s successively. The resulting matrix has rank 19 and its nullspace is spanned by a vector with small rational coefficients. Multiplying by the lcm of the denominators we get the following cartesian equation:

$$224\,y^3 - 7056\,y^2 x + 33168\,y^2 z + 60426\,yx^2 - 562500\,yxz + 1322088\,yz^2 - 156195\,x^3$$
$$+2188998\,x^2 z - 10175796\,xz^2 + 15631624\,z^3 = 0.$$

6 Conclusion

We present a new method for doing implicitization of curves, surfaces and hypersurfaces, using essentially linear algebra. The method works for polynomial, rational and trigonometric parametric equations. The method also applies to monoparametric families of parametric curves, surfaces and hypersurfaces, with a small amount of extra work. The method is quite efficient due to the fact that it does not use Gröbner bases or multivariate factorization computations. The efficiency of the method can be improved by taking into account the special structure of the matrices involved in the computation.

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