Matrix Padé Fractions

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ABSTRACT

For matrix power series with coefficients over a field, the notion of a matrix power series remainder sequence and its corresponding cofactor sequence are introduced and developed. An algorithm for constructing these sequences is presented.

It is shown that the cofactor sequence yields directly a sequence of Padé fractions for a matrix power series represented as a quotient $B(z)^{-1}A(z)$. When $B(z)^{-1}A(z)$ is normal, the complexity of the algorithm for computing a Padé fraction of type (m,n) is $O(p^3(m+n)^2)$, where p is the order of the matrices A(z) and B(z).

For power series which are abnormal, for a given (m,n), Padé fractions may not exist. However, it is shown that a generalized notion of Padé fraction, the Padé form, introduced in this paper does always exist and can be computed by the algorithm. In the abnormal case, the algorithm can reach a complexity of $O(p^3(m+n)^3)$, depending on the nature of the abnormalities. In the special case of a scalar power series, however, the algorithm complexity is $O((m+n)^2)$, even in the abnormal case.

1. Introduction.

Let

$$A(z) = \sum_{i=0}^{\infty} a_i z^i, \tag{1.1}$$

where a_i, i = 0, ... , is a p x p matrix with coefficients from a field K, be a formal power series. Loosely speaking, a matrix Padé approximant of A(z) is an expression of the form $U(z) \cdot V(z)^{-1}$, or $V(z)^{-1} \cdot U(z)$, where U(z) and V(z) are matrix polynomials of degree at most m and n, respectively, whose expansion agrees with A(z) up to and including the term z^{m+n}.

The definition of a Padé approximant can be made more formal in a variety of ways. For example, Rissanen [9] restricts V(z) to be a scalar polynomial and allows U(z) to be a p x q matrix. Typically, however, U(z) and V(z) are p x p polynomial matrices, and V(z) is further restricted by the condition that the constant term, V(0), is invertible (c.f., Bose and Basu[1], Bultheel[3], and Starkand[12]). In this paper, we call such approximants matrix Padé fractions, which is consistent with the scalar (p=1) case (c.f., Gragg[7]).

Definition 2.1: The pair of matrix polynomials (U(z),V(z)) is defined to be a **Right Matrix Padé Form** (RMPFo) of type (m,n) for the pair (A(z),B(z)) if

I. $\partial(U(z)) \leq m, \quad \partial(V(z)) \leq n,$

$$II. \quad A(z)\cdot V(z) + B(z)\cdot U(z) = z^{m+n+1}W(z) \tag{2.3}$$

where W(z) is a formal matrix power series, and

III. The columns of $V(z)$ are linearly independent over the field K. ■

The matrix polynomials U(z), V(z), and W(z) are usually called the right numerator, denominator, and residual (all of type (m,n)), respectively. Note that when B(z) = -I, Definition 2.1 corresponds to the definition of Padé form for a single matrix power series A(z) given in Labahn[8].

For ease of discussion, we use the following notation. For any matrix polynomial

$$U(z) = u_0 + u_1 z + \cdots + u_k z^k , \tag{2.5}$$

we write U (i.e., the same symbol but without the z variable) to mean the p(k+1) by p vector of matrix coefficients

$$U = \left[u_0, u_1, \cdots, u_k\right]^t , \tag{2.6}$$

where the transpose is at the symbolic level.

Let

$$S_{m,n} = \left[\begin{array}{cccc|cccc} a_0 & & & & b_0 & & & \\ & \cdot & & & & \cdot & & \\ \cdot & & \cdot & & \cdot & & \cdot & b_0 \\ \cdot & & & a_0 & \cdot & & & \cdot \\ & & & \cdot & \cdot & & & \cdot \\ & & & \cdot & & & & \cdot \\ a_{m+n} & \cdot & \cdot & a_m & b_{m+n} & \cdot & \cdot & b_n \end{array}\right] \tag{2.7}$$

denote a Sylvester matrix for A(z) and B(z) of type (m,n). Then equation (2.3) can be written as

$$S_{m,n}\cdot \begin{bmatrix} V \\ U \end{bmatrix} = 0. \tag{2.8}$$

Theorem 2.2: (Existence of Matrix Padé Forms) For any pair of power series (A(z), B(z)) and any pair of nonzero integers (m,n), there exists a RMPFo of type (m,n).

One case when the RMPFo is unique is given by

Definition 2.3. A pair (U(z),V(z)) of p x p matrix polynomials is said to be a **Right Matrix Padé Fraction** (RMPFr) of type (m,n) for the pair (A(z),B(z)) if

I. (U(z),V(z)) is a RMPFo of type (m,n) for (A(z),B(z)), and

II. The constant term, V(0), of the denominator is an invertible matrix. ■

For a particular m and n, however, matrix Padé fractions need not exist. Therefore, in this paper, we introduce the notion of a matrix Padé form, in which the condition of invertibility of V(0) is relaxed. The definition is a generalization of a similar one given for the scalar case (c.f., Gragg[7]). It is shown that matrix Padé forms always exist, but that they may not be unique. In general, matrix Padé forms need not have an invertible denominator, V(z). However, for m and n given, by obtaining a basis for all the Padé forms, we are also able to construct a matrix Padé form with an invertible V(z), in the case that one does exist.

In the one dimensional case, some algorithms that calculate Padé approximants for normal power series (Gragg[7]) include the ϵ-algorithm of Wynn, the η-algorithm of Bauer, and the Q-D algorithm of Rutishauser. Algorithms that are successful in the degenerate non-normal case are given by Brent et al[2], Bultheel[4], Rissanen[11], and Cabay and Choi[5].

The matrix case parallels the scalar situation in that most algorithms are restricted to normal power series. Algorithms that require the normality condition include those of Bultheel[3], Bose and Basu[1], Starkand[12], and Rissanen[10]. An algorithm that calculates Padé approximants in a non-normal case is given by Labahn[8]. However, in this algorithm there are still strict conditions that need to be satisfied by the power series before Padé approximants can be calculated.

The primary contribution of this paper is an algorithm, MPADE, for computing matrix Padé forms for a matrix power series. Central to the development of MPADE are the notions of a matrix power series remainder sequence and the corresponding cofactor sequence, which are introduced in section 4. These are generalizations of notions developed by Cabay and Kossowski[6] for power series over an integral domain. The cofactor sequence computed by MPADE yields a sequence of matrix Padé fractions along a specific off-diagonal path of the Padé table for A(z).

Unlike other algorithms, there are no restrictions placed on the power series in order that MPADE succeed. For normal power series, the complexity of MPADE is $O(p^3 \cdot (m+n)^2)$ operations in K. This is the same complexity as some of the algorithms proposed by Bultheel[3], Bose and Basu [1], Starkand [12], and Rissanen[10]. In the abnormal case, the complexity of the algorithm can reach $O(p^3 \cdot (n+n)^3)$ operations in K, depending on the nature of the abnormalities.

2. Matrix Padé Forms and Matrix Padé Fractions.

Let A(z) and B(z) be formal power series

$$A(z) = \sum_{i=0}^{\infty} a_i z^i \ , \ B(z) = \sum_{i=0}^{\infty} b_i z^i \tag{2.1}$$

with coefficients from the ring of p x p matrices over some field K. Throughout this paper it is assumed that the leading coefficient, b_0, of B(z) is an invertible matrix. For non-negative integers m and n, let

$$U(z) = \sum_{i=0}^{m} u_i z^i \ , \ V(z) = \sum_{i=0}^{n} v_i z^i \tag{2.2}$$

denote p x p matrix polynomials.

Condition II ensures that the denominator, V(z), is an invertible matrix polynomial.

The problem with Padé fractions, as mentioned in the previously , is that they do not always exist. However, let

$$T_{m,n} = \left[\begin{array}{cccc|cccc} a_0 & & & & b_0 & & & \\ \cdot & \cdot & & & \cdot & \cdot & & \\ \cdot & & \cdot & & & & & b_0 \\ \cdot & & & a_0 & \cdot & & & \cdot \\ & & & \cdot & & & & \cdot \\ & & & \cdot & & & & \\ a_{m+n-1} & \cdot & \cdot & a_m & b_{m+n-1} & \cdot & \cdot & b_n \end{array}\right] \tag{2.9}$$

and define

$$d_{m,n} = \begin{cases} 1, & m = 0 , n = 0, \\ det(T_{m,n}), & otherwise. \end{cases} \tag{2.10}$$

Then, a sufficient condition for the existence of a RMPFr is given by

Theorem 2.4. If $d_{m,n} \neq 0$, then every RMPFo of type (m,n) is an RMPFr of type (m,n). In addition, a RMPFr of type (m,n) is unique up to multiplication on the right by a nonsingular p x p matrix having coefficients from the field K. ∎

In the next section we also require

Theorem 2.5. Let A(z) and B(z) be given by (2.1). If m and n are positive integers such that $d_{m,n} \neq 0$, then RMPFo's (P(z),Q(z)) of type (m-1,n-1) for (A(z),B(z)) are unique up to multiplication of P(z) and Q(z) on the right by a nonsingular matrix from K. In addition, the leading term R(0) of the residual in condition II for RMPFo's,

$$A(z)\cdot Q(z) + B(z)\cdot P(z) = z^{m+n-1}R(z) \,, \tag{2.11}$$

is a nonsingular matrix. ∎

3. Matrix Power Series Remainder Sequences.

We define a **Right Matrix Padé Table** for (A(z), B(z)) to be any infinite two-dimensional collection of RMPFo's of type (m,n) for (A(z),B(z)) with m = 0, 1, ... and n = 0, 1, ... It is assumed that there is precisely one entry (i.e., one RMPFo) assigned to each position in the table. From Theorem 2.2, it follows that a right matrix Padé table exists for any given (A(z), B(z)). However, the table is not unique, because RMPFo's are not unique. This is unlike the definition of a Padé table for scalar power series (c.f. Gragg[7]), since here a Padé table consists of a collection of Padé fractions, which are unique.

A matrix power series pair (A(z),B(z)) is said to be **normal** (c.f., Bultheel[3]) if $d_{m,n} \neq 0$ for all m,n. For normal power series, it follows from Theorem 3.2 that every entry in the right matrix Padé table is a RMPFr. Consequently, from condition II in Definition 2.3 of RMPFr's, a right matrix

Padé table for normal power series may be made unique by insisting that the constant term, V(0), in the denominator of any Padé fraction be the identity matrix.

Following the convention used in the scalar case (c.f., Gragg[7]), we also define

$$(U(z),\, V(z)) = (z^m I,\, 0) \quad \text{for} \quad m \geq -1, \;\; n = -1, \tag{3.1}$$

and

$$(U(z),\, V(z)) = (0,\, z^n I) \quad \text{for} \quad m = -1, \;\; n \geq 0. \tag{3.2}$$

A right matrix Padé table appended with (3.1) and (3.2) is called an **extended right matrix Padé table** (c.f., Gragg[7]). The use of an extended table is strictly for initialization purposes. The entries given by (3.1) and (3.2) are not right matrix Padé forms (indeed the (-1,-1) entry is not even a matrix polynomial). However they do satisfy property II of Definition 2.1. For example, for $m \geq -1$ and $n = -1$, we have that

$$A(z)V(z) + B(z)U(z) = z^{m+n+1}W(z) \tag{3.3}$$

with

$$W(z) = B(z); \tag{3.4}$$

while, for $m = -1$ and $n \geq 0$, we have (3.3) with

$$W(z) = A(z). \tag{3.5}$$

Given the power series (2.1) and any non-negative integers m and n, we introduce a sequence of points

$$(m_0,n_0),\;\; (m_1,n_1),\;\; (m_2,n_2),\;\; \cdots \tag{3.6}$$

in the extended right matrix Padé table by setting

$$(m_0,n_0) = \begin{cases} (m-n-1,-1) & \text{for} \quad m \geq n \\ (-1,n-m-1) & \text{for} \quad m < n \end{cases} \tag{3.7}$$

and

$$(m_{i+1},n_{i+1}) = (m_i + s_i\,, n_i + s_i\,),\;\; i = 0, 1, 2, \;\cdots\;, \tag{3.8}$$

where $s_i \geq 1$. Observe that

$$m_i - n_i = m - n\;,\;\; i = 0, 1, 2, \;\cdots\;, \tag{3.9}$$

and consequently the sequence (3.6) lies along the m-n off-diagonal path of the extended right matrix Padé table. In (3.8), the s_i are selected so that

$$d_{m_{i+1},n_{i+1}} \neq 0 \tag{3.10}$$

and

$$d_{(m_i+j),(n_i+j)} = 0\ , \tag{3.11}$$

for $j = 1, 2, \cdots, s_i-1$.

For i = 1, 2, ..., let $(U_i(z),V_i(z))$ be the unique RMPFr (c.f., Theorem 3.2) of type (m_i,n_i) for (A(z),B(z)). Thus $\left[V_i\ , U_i\ \right]^t$ satisfies

$$S_{m_i,n_i} \cdot \begin{bmatrix} V_i \\ U_i \end{bmatrix} = 0 \tag{3.12}$$

and, according to (2.3), there exists a matrix power series $W_i(z)$ such that

$$A(z)\cdot V_i(z) + B(z)\cdot U_i(z) = z^{m_i + n_i + 1} W_i(z). \tag{3.13}$$

Generalizing the notions of Cabay and Kossowski[6], we introduce

Definition 3.1. The sequence

$$\left\{W_i(z)\right\},\ i = 1, 2, \cdots, \tag{3.14}$$

is called the **Power Series Remainder Sequence** for the pair (A(z),B(z)). The sequence of pairs

$$\left\{(U_i(z),V_i(z))\right\},\ i = 1, 2, \cdots, \tag{3.15}$$

is called the corresponding **cofactor sequence.** The integer pairs $\{(m_i,\ n_i)\}$ are called **nonsingular nodes** along the m - n off-diagonal path of the extended right matrix Padé table for (A(z),B(z)). ∎

We note that each term of a power series remainder sequence is unique up to multiplication on the right by a nonsingular matrix. This is also true for each term of the corresponding cofactor sequence.

Initially, when $m \geq n$, observe that $m_1 = m - n$ and $n_1 = 0$ (i.e., $s_0 = 1$), because in (3.2) the nonsingularity of b_0 implies that $d_{(m-n),0} \neq 0$. Thus, $V_1(z)$ is some arbitrary nonsingular matrix from K and, using (3.12), $U_1(z)$ can be obtained by solving

$$\begin{bmatrix} b_0 & & \\ \cdot & \cdot & \cdot \\ b_{m_1} & \cdots & b_0 \end{bmatrix} U_1 = -\begin{bmatrix} a_0 \\ \vdots \\ a_{m_1} \end{bmatrix} V_1\,. \tag{3.16}$$

That is, $U_1(z)$ can be obtained by multiplying the first m_1+1 terms of the quotient power series $B^{-1}(z)\cdot A(z)$ on the right by $-V_1(z)$.

Initially, when $m < n$, depending on a_0 there are two cases to consider. The simple case, when $det(a_0) \neq 0$, yields

$$d_{0,(n-m)} = \det \begin{bmatrix} a_0 & & \\ & \cdot & \\ & & \cdot \\ a_{n-m-1} & \cdot\ \cdot & a_0 \end{bmatrix} \neq 0 . \tag{3.17}$$

Thus, $s_0 = 1$, $m_1 = 0$, and $n_1 = n-m$. Then, the RMPFr $(U_1(z),V_1(z))$ of type (m_1,n_1) is determined by setting $U_1(z)$ to be an arbitrary nonsingular matrix from K and then solving

$$\begin{bmatrix} a_0 & & \\ \cdot & \cdot & \\ \cdot & & \cdot \\ a_{n_1} & \cdot\ \cdot & a_0 \end{bmatrix} V_1 = -\begin{bmatrix} b_0 \\ \cdot \\ \cdot \\ b_{n_1} \end{bmatrix} U_1 . \tag{3.18}$$

That is, when $m < n$ and $det(a_0) \neq 0$, $V_1(z)$ can be obtained by multiplying the first n_1+1 terms of the quotient power series $A^{-1}(z) \cdot B(z)$ on the right by $-U_1(z)$.

When $m < n$ and $det(a_0) = 0$, we must first determine the smallest positive integer s_0 (i.e., the smallest $m_1 = m_0+s_0$ and $n_1 = n_0+s_0$) so that $d_{m_1,n_1} \neq 0$. Notice that we must have $s_0 \geq n-m+1$. Once s_0 has been obtained, then $(U_1(z), V_1(z))$ is obtained by solving

$$S_{m_1,n_1} \cdot \begin{bmatrix} V_1 \\ U_1 \end{bmatrix} = 0 . \tag{3.19}$$

In section 5, we give an algorithm which computes a RMPFo of type (m,n) for (A(z),B(z)) by performing a sequence of the above types of initializations (albeit, each for different power series).

When the power series pair (A(z),B(z)) is normal, only the initializations corresponding to (3.16) and (3.18) are required. Thus, for normal power series $s_i = 1$ for all $i \geq 1$, and the algorithm reduces to a sequence of truncated power series divisions.

There are also some non-normal power series that share this property. For each pair of integers m and n, let $r_{m,n}$ be the rank of the matrix $T_{m,n}$. Then normality is equivalent to

$$r_{m,n} = (m+n) \cdot p \tag{3.20}$$

for all m and n. A matrix power series pair (A(z), B(z)) is said to be **nearly-normal** (c.f., Labahn[8]) if, for all integers m and n,

$$r_{m,n} = k_{m,n} \cdot p \tag{3.21}$$

for some integer $k_{m,n}$. Clearly, every normal power series is also a nearly-normal power series. In addition, all scalar power series are nearly-normal.

For a nearly-normal power series pair (A(z),B(z)) it is easy to see that when a_0 is singular, then $a_0 = 0$. This follows from the observation that the rank of a_0 is just $r_{0,1}$, which, if it is not p, must be zero. Also, if $a_0 = \cdots = a_{k-1} = 0$ and $a_k \neq 0$, then a_k must be a nonsingular matrix for similar reasons. When $k > m$ this implies that there are no nonsingular nodes along the m - n off-diagonal path before and including the node (m,n). Otherwise, when $k \leq m$, the initialization (3.19) becomes

$$\begin{bmatrix} 0 & & & | & b_0 & & \\ & & & | & & \cdot & \\ a_{m_1} & & & | & \cdot & & \cdot \\ \cdot & \cdot & & | & \cdot & & b_0 \\ \cdot & & \cdot & | & \cdot & & \\ a_{m_1+n_1} & \cdot\ \cdot & a_{m_1} & | & b_{m_1+n_1} & \cdot\ \cdot & b_{n_1} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ \\ U_1 \end{bmatrix} = 0, \tag{3.22}$$

where $s_0 = k+1$, $m_1 = k$ and $n_1 = n-m+k$. Consequently the RMPFr $(U_1(z),V_1(z))$ of type (m_1,n_1) is obtained from (3.22) first by setting $U_1(z) = z^{m_1}\cdot U$, where U is any nonsingular matrix from K. Then, $V_1(z)$ is obtained by multiplying the first n_1+1 terms of the quotient power series $(z^{-m_1}\cdot A(z))^{-1}\cdot B(z)$ on the right by -U. Thus, also for nearly-normal power series (and therefore also for all scalar power series), all initializations reduce to truncated power series divisions.

Corresponding to the Power Series Remainder Sequence, we introduce

Definition 3.2. The sequence

$$\left\{ (P_i(z),Q_i(z)) \right\}, \quad i = 1, 2, \cdots , \tag{3.23}$$

where $(P_i(z),Q_i(z))$ is the $(m_i-1,\, n_i-1)$ entry in the extended matrix Padé table for (A(z),B(z)), is called a **predecessor sequence** of the power series remainder sequence. ■

Theorem 3.3: For i = 1, 2, ... , the predecessors $(P_i(z),Q_i(z))$ are unique up to right multiplication by a nonsingular matrix from K. In addition, the leading term of the residual, $R_i(0)$, is nonsingular. ■

The main result of this section is

Theorem 3.4. For any positive integer k, (k - 1, k) is a nonsingular node in the Padé table for $(W_i(z),R_i(z))$ if and only if $(m_i+k,\, n_i+k)$ is a nonsingular node in the Padé table for (A(z), B(z)). ■

Theorem 3.4 allows us to calculate nonsingular nodes of a pair of power series by calculating nonsingular nodes of the residual pair of power series. This gives us an iterative method of calculating nonsingular nodes.

Theorem 3.5: The cofactor and predecessor sequences for (A(z),B(z)) satisfies

$$\begin{bmatrix} U_{i+1}(z) & P_{i+1}(z) \\ V_{i+1}(z) & Q_{i+1}(z) \end{bmatrix} = \begin{bmatrix} U_i(z) & P_i(z) \\ V_i(z) & Q_i(z) \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & z^2\cdot I \end{bmatrix} \cdot \begin{bmatrix} V'(z) & Q'(z) \\ U'(z) & P'(z) \end{bmatrix} \tag{3.25}$$

where $(U'(z),V'(z))$ is the RMPFr of type $(s_i-1,\, s_i)$ for $(W_i(z),R_i(z))$, and where $(P'(z),Q'(z))$ is its predecessor. ■

4. The Algorithm:

Given non-negative integers m and n, the algorithm MPADE below makes use of Theorem 3.5 to compute the cofactor and predecessor sequences (3.10) and (3.18), respectively. Thus, intermediate results available from MPADE include those RMPFr's $(U_i(z),V_i(z))$ for (A(z),B(z)) at all the non-singular nodes (m_i, n_i), i=1,2, ... , k-1, smaller than (m,n), along the off-diagonal path m_i-n_i = m-n. The output gives results associated with the final node (m_k,n_k). If (m,n) is also a nonsingular node, then the output $(U_k(z),V_k(z))$ is a RMPFr of type (m,n) for (A(z),B(z)), and $(P_k(z),Q_k(z))$ is a RMPFo of type (m-1,n-1). If (m,n) is a singular node, then the output $(U_k(z),V_k(z))$ is simply a RMPFo of type (m,n) for (A(z),B(z)), and now $(P_k(z),Q_k(z))$ is set to be the RMPFr of type (m_{k-1}, n_{k-1}).

Note that, when (m,n) is not a nonsingular node, a simple modification of MPADE allows the computation of all RMPFo's of type (m,n) for (A(z),B(z)). It is only necessary to arrange to compute q columns of $[V'_k , U'_k]$, rather than p, in order to form a basis for the solution space of the equation in step 3.1 of MPADE. From this basis, it is then possible to construct a p x p matrix V(z), and a corresponding U(z), for which (U(z),V(z)) is a RMPFo of type (m,n) for (A(z),B(z)) and has the property that V(z) is an invertible matrix, assuming such a RMPFo exists. This enhancement is not included in MPADE primarily to simplify the presentation of the algorithm.

ALGORITHM (MPADE):

Step 1: # Initialization #

If $m \geq n$

then set

1.1) $i \leftarrow 1$

1.2) $s_0 \leftarrow m - n$

1.3) $$\begin{bmatrix} m_1 \\ n_1 \end{bmatrix} = \begin{bmatrix} s_0 \\ 0 \end{bmatrix}$$

1.4) $$\begin{bmatrix} U_1(z) & P_1(z) \\ V_1(z) & Q_1(z) \end{bmatrix} = \begin{bmatrix} -B(z)^{-1} \cdot A(z) \bmod z^{s_0+1} & z^{s_0-1} \cdot I \\ I & 0 \end{bmatrix}$$

else set

1.5) $i \leftarrow 0$

1.6) $$\begin{bmatrix} m_0 \\ n_0 \end{bmatrix} = \begin{bmatrix} m-n \\ 0 \end{bmatrix}$$

1.7) $$\begin{bmatrix} U_0(z) & P_0(z) \\ V_0(z) & Q_0(z) \end{bmatrix} = \begin{bmatrix} 0 & z^{m-n-1} \cdot I \\ I & 0 \end{bmatrix}$$

Step 2: # Search for next nonsingular node #

2.1) $s_i \leftarrow 0$

2.2) d ← 0

2.3) Do while $n_i + s_i < n$ and d = 0

2.4) Set $s_i \leftarrow s_i + 1$

2.5) Compute the residual $W_i(z)$ such that

$$(A(z) \cdot V_i(z) + B(z) \cdot U_i(z)) \bmod z^{m_i+n_i+2s_i+1} = z^{m_i+n_i+1} \cdot W_i(z)$$

2.6) Compute the residual $R_i(z)$ such that

$$(A(z) \cdot Q_i(z) + B(z) \cdot P_i(z)) \bmod z^{m_i+n_i+2s_i-1} = z^{m_i+n_i-1} \cdot R_i(z)$$

2.7) Compute

$$d = det(T'_{(s_i-1),s_i}).$$

determined from the power series $W_i(z)$ and $R_i(z)$

2.8) End do

Step 3: # Compute RMPFr for residuals #

3.1) Solve

$$S'_{(s_i-1),s_i} \cdot \begin{bmatrix} V' \\ U' \end{bmatrix} = 0,$$

where S' is the Sylvester matrix determined from $W_i(z)$ and $R_i(z)$

Step 4: # Compute predecessor for residuals #

4.1) If $s_i > 1$ and $d \neq 0$,

then solve

$$S'_{(s_i-2),(s_i-1)} \cdot \begin{bmatrix} Q' \\ P' \end{bmatrix} = 0 ,$$

where again S' is the Sylvester matrix determined from $W_i(z)$ and $R_i(z)$

else set

4.2) $$\begin{bmatrix} Q'(z) \\ P'(z) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Step 5: # Advance along off-diagonal for Padé fractions #

5.1) $m_{i+1} \leftarrow m_i + s_i$

5.2) $n_{i+1} \leftarrow n_i + s_i$

5.3) $$\begin{bmatrix} U_{i+1}(z) & P_{i+1}(z) \\ V_{i+1}(z) & Q_{i+1}(z) \end{bmatrix} \leftarrow \begin{bmatrix} U_i(z) & P_i(z) \\ V_i(z) & Q_i(z) \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & z^2 \cdot I \end{bmatrix} \cdot \begin{bmatrix} V'(z) & Q'(z) \\ U'(z) & P'(z) \end{bmatrix}$$

5.4) i ← i + 1

Step 6: # termination test #

If $n_i < n$
then go to step 2
Else $k \leftarrow i$

$$\text{return}(\begin{bmatrix} U_k(z) & P_k(z) \\ V_k(z) & Q_k(z) \end{bmatrix})$$

■

5. Complexity of MPADE Algorithm

In assessing the costs of MPADE, it is assumed that classical algorithms are used for the multiplication of polynomials. Only the more costly steps are considered. For these steps, Table 5.1 below provides crude upper bounds on the number of multiplications in K performed during the i-th pass through MPADE.

Step	Bound on Number of Multiplications
2.5	$2p^3(m_i+n_i+2)s_i$
2.6	$2p^3(m_i+n_i+2)s_i$
2.7	$8p^3(s_i-1)^3/3$
3.1	$2p^3s_i^2$
4.1	$2p^3s_i^2$
5.3	$4p^3(m_i+n_i+2)(s_i+1)$

Table 5.1 : Bounds on Operations per Step

In step 2.7 of MPADE, it is assumed that the Gaussian elimination method is used to obtain the LU decomposition of $T_{(s_i-1),s_i}$. In addition, it is assumed that Gaussian elimination is accompanied with bordering techniques. Thus, as s_i increases by 1 in step 2.4, the results of the previous pass through the **while** loop are used to achieve the current LU decomposition. The bound for step 2.7 in Table 5.1 assumes we do not take any advantage of the special nature of $T_{(s_i-1),s_i}$.

For step 3.1, it is assumed that the LU decomposition of $T_{(s_i-1),s_i}$ from step 2.7, is used to simplify the triangulation of $S_{(s_i-1),s_i}$. The solution $[V', U']$ is obtained finally by solving this triangularized $S_{(s_i-1),s_i}$. Similar observations apply to step 4.1.

An upper bound for the number of multiplications in K required by MPADE is obtained by summing the costs in Table 5.1 for i=0,1, ... ,k. We use the fact that

$$\sum_{i=0}^{k} s_i = m \text{ , if } m \geq n \text{ , and } \sum_{i=0}^{k} s_i = n \text{ , if } m < n \text{ .} \tag{5.8}$$

In addition,

$$\sum_{i=0}^{k} m_i^{\alpha} s_i^{\beta} \leq m^{\alpha+\beta} \text{ and } \sum_{i=0}^{k} n_i^{\alpha} s_i^{\beta} \leq n^{\alpha+\beta} \text{ .} \tag{5.9}$$

Then, step 2.7 has a complexity of $O(p^3(m+n)^3)$ and the remaining steps a complexity of $O(p^3(m+n)^2)$, at worst. When (A(z),B(z)) is normal (i.e., $s_i = 1$ for all i), then the cost of step 2.7 is zero, since $T_{(s_i-1),s_i}$ is already in triangular form, and the complexity of MPADE then reduces to $O(p^3(m+n)^2)$. This is also the case when (A(z),B(z)) is nearly-normal. In this case s_i is often larger than one, but the matrix $T_{(s_i-1),s_i}$ is always in triangular form and so again the complexity is $O(p^3(m+n)^2)$. In particular, in the scalar case the complexity of MPADE is $O((m+n)^2)$.

When the power series is neither normal nor nearly-normal, MPADE still provides significant savings even in the case where most intermediate nodes are singular. For example, if only the middle node (n/2,n/2) along the main diagonal is nonsingular, then MPADE has a complexity of $8(n/2)^3/3 + 8(n/2)^3/3 = 2n^3/3$. This is a saving of a factor of 4 over the simple use of Gaussian elimination. Algorithms requiring normality, on the other hand, break down when even one intermediate node is singular.

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