Walking Streets Faster*

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Abstract

A fundamental problem in robotics is to compute a path for a robot from its current location to a given goal. In this paper we consider the problem of a robot equipped with an on-board vision system searching for a goal g in an unknown environment.

We assume that the robot is located at a point s in a polygon that belongs to the well investigated class of polygons called *streets*. A *street* is a simple polygon where s and g are located on the polygon boundary and the part of the polygon boundary from s to g is weakly visible to the part from g to s and vice versa.

Our aim is to minimize the ratio of the length of the path traveled by the robot to the length of the shortest path from s to g. In analogy to on-line algorithms this value is called the competitive ratio. We present two strategies. Our first strategy, *continuous lad*, extends the strategy *lad* which minimizes the Local Absolute Detour. We show that this extension results in a 2.03-competitive strategy, which significantly improves the best known bound of 4.44 for this class of strategies. Secondly, and most importantly, we present a hybrid strategy consisting of *continuous lad* and the strategy *Move-in-Quadrant*. We show that this combination of strategies achieves a competitive ratio of 1.73 which about halves the gap between the known $\sqrt{2}$ lower bound for this problem and the previously best known competitive ratio of 2.05.

1 Introduction

Finding a path from a starting location to a goal within a given scene is an important problem in robotics. A natural and realistic setting is to assume that the robot has only a partial knowledge of its surroundings and that the amount of information available to the robot increases as it discovers its surroundings on its travels. For this purpose, the robot is equipped with an on-board vision system that provides the visibility map of its local environment. The robot uses this information to devise a search path for a visually identifiable goal located outside the current visibility region. The quality of a search strategy is then evaluated under the framework of competitive analysis for on-line searches, as introduced by Sleator and Tarjan [13]. A search strategy is called *c-competitive* if the path traveled by the robot to find the goal is at most c times longer than a shortest path. The parameter c is called the *competitive ratio* of the strategy.

As can easily be seen, there is no strategy with a competitive ratio of $o(n)$ for scenes with arbitrary obstacles having a total of n vertices [2] even if we restrict ourselves to searching in a simple polygon. Therefore, the on-line search problem has been studied previously in various contexts where the geometry of the obstacles is restricted [1, 2, 3, 9, 7, 10, 12].

Klein introduced the notion of a *street* as the first class of polygons which allow search strategies with a constant competitive ratio even when the location of the goal is unknown [6]. In a street, the starting point s and the goal g are located on the boundary of the polygon and the two polygonal chains from s to g are mutually weakly visible. Klein presents the strategy *lad* for searching in streets which is based on the idea of minimizing the Local Absolute Detour. He shows an upper bound on its competitive ratio of $1+3/2\pi$ ($\sim$ 5.71), later improved to $1+\pi/2+\sqrt{1+\pi^2/4}$ (~ 4.44) by Icking [5].

A strategy based on a different approach was presented by Kleinberg [7]. His strategy for searching in streets can be shown to have a competitive ratio of $2\sqrt{2}$ with a very simple analysis. A further improvement using ideas similar to Kleinberg's achieves a competitive ratio of $\sqrt{1+(1+\pi/4)^2}$ (~ 2.05) [8], however the analysis is significantly more complex.

As Figure 1 shows, all strategies must have at least a $\sqrt{2}$ competitive ratio. Here, if a strategy moves to the left or right before seeing g, then g can be

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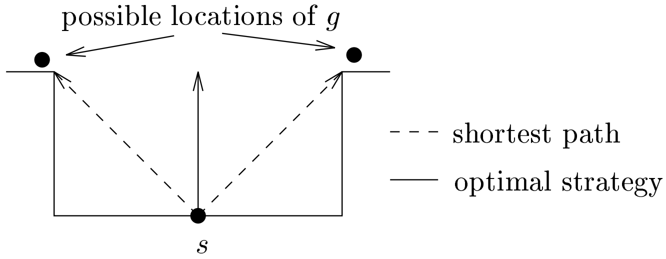

Figure 1: A lower bound for searching in rectilinear streets.

placed on the opposite side, thus forcing the robot to travel more than $\sqrt{2}$ times the diagonal. This is the only known lower bound even for arbitrarily oriented streets.

In this paper we present two strategies to traverse a street, one of which is an extension of the original approach presented by Klein. The first strategy presented, *continuous lad*, is shown to have a ~ 2.03 competitive ratio, which significantly improves the best known bound of 4.44 for this class of strategies. We then combine *continuous lad* with the strategy *Move-in-Quadrant* presented in [8], resulting in a hybrid strategy with a competitive ratio of 1.73. This new strategy reduces in more than half the gap between the known $\sqrt{2}$ lower bound for this problem and the previously best known strategy of ~ 2.05.

The paper is organized as follows. In Section 2 we introduce the basic geometric concepts necessary for the rest of the paper. We also introduce a "High Level Strategy" as proposed by Klein [6]. We state some results about search strategies that follow this High Level Strategy in Section 3 and present the new strategy, *continuous lad*, and its analysis. In Section 4 we provide an new analysis of the strategy *Move-in-Quadrant* [8] and show how to combine *continuous lad* with *Move-in-Quadrant* optimally to obtain a strategy with a performance guarantee of 1.73.

2 Preliminaries

Since we deal with point sets in the plane $\mathbb{E}^2$, we need the standard definitions of distance, norm, angle etc. for points. If p, q, and r are three points in the plane, then we denote

(i) the L_2-distance between p and q by $d(p,q)$,

(ii) the line segment between p and q by $\overline{pq}$, and

(iii) the counterclockwise angle between the line segment $\overline{qp}$ and the line segment $\overline{qr}$ at q by $\angle pqr$.

If $\mathcal{P}$ is a path in $\mathbb{E}^2$, we denote its length by $\lambda(\mathcal{P})$. Furthermore, if p and q are two points on $\mathcal{P}$, then we denote the part of $\mathcal{P}$ from p to q by $\mathcal{P}(p,q)$.

A *simple polygon* is a simple, closed curve that consists of the concatenation of line segments, called the *edges* of the polygon, such that no two consecutive edges are collinear. The end points of the edges are called the *vertices* of the polygon.

We consider a simple polygon P in the plane with n vertices and a robot inside P which is located at a start point s on the boundary of P. The robot has to find a path from s to the goal g. We denote the shortest path from s to g by $sp(s,g)$.

The search of the robot is aided by simple vision (i.e. we assume that the robot knows the visibility polygon of its current location). Furthermore, the robot retains all the information seen so far (in memory) and knows its starting and current position. We are, in particular, concerned with a special class of polygons called *streets* first introduced by Klein [6].

Definition 2.1 [6] *Let P be a simple polygon with two distinguished vertices, s and g, and let L and R denote the clockwise and counterclockwise, resp., oriented boundary chains leading from s to g. If L and R are mutually weakly visible, i.e. if each point of L sees at least one point of R and vice versa, then (P,s,g) is called a* street.

The only available information to the robot is its *visiblity polygon.*

Definition 2.2 *Let P be a street with start point s and goal g. If p is a point of P, then the* visibility polygon *of p is the set of all points in P that are seen by p. It is denoted by $V(p)$.*

A window *of $V(p)$ is an edge of $V(p)$ that does not belong to the boundary of P (see Figure 2a).*

A window w splits P into a number of subpolygons $P_1,\ldots,P_k$ one of which contains $V(p)$. We denote the union of the subpolygons that do not contain $V(p)$ by P_w.

The end point of a window w that is closer to p is called the *entrance point* of w. We assume that a window w has the orientation of the ray from p to entrance point of w. We say a window w is a *left window* if P_w is locally to the left of w w.r.t. the given orientation of w. A *right window* is defined similarly.

Let p be the current location of the robot and $\mathcal{P}_{sp}$ the path the robot followed from s to p. We assume that the robot knows the part of P that can be seen from $\mathcal{P}_{sp}$, i.e. the robot maintains the polygon $V(\mathcal{P}_{sp}) = \bigcup_{q\in\mathcal{P}_{sp}} V(q)$. We say a window w of $V(p)$ is a *true window* w.r.t. $\mathcal{P}_{sp}$ if P_w is not contained in $V(\mathcal{P}_{sp})$. We say two (true) windows w_1 and

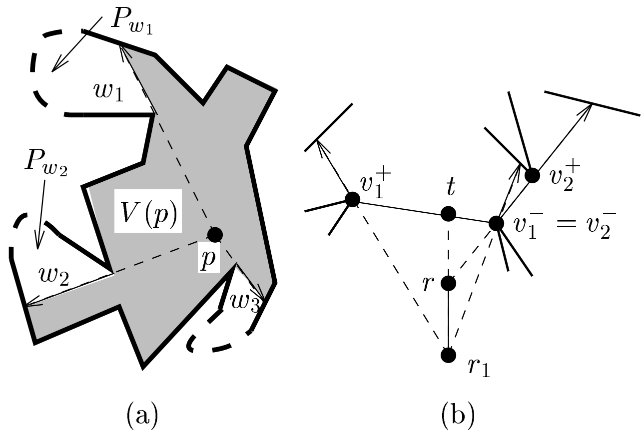

Figure 2: (a): The visibility polygon $V(p)$ of p with left window w_1 and right windows w_2 and w_3. (b): As the robot moves to t the left extreme entrance point "jumps" from v_1^+ to v_2^+ and the robot moves directly to v_1^-.

w_2 are *clockwise consecutive* if the clockwise oriented polygonal chain of $V(p)$ between w_1 and w_2 does not contain a (true) window different from w_1 and w_2. *Counterclockwise consecutive* is defined analogously.

If w_0 is the window of $V(p)$ that is intersected the first time by $\mathcal{P}_{sp}$, then it can be shown that all left true windows are clockwise consecutive and all right true windows are counterclockwise consecutive from w_0 [6, 7, 8]. Hence, if left true windows exist, then there is a clockwise-most left true window in $V(p)$ which we call the *left extreme true window* and denote by w^+. The *right extreme true window* w^- is defined similarly. The entrance point v^+ (v^-) of w^+ (w^-) is called the *left* (*right*) *extreme entrance point of* $V(p)$. It can be easily shown that g is contained in either P_{w^+} or P_{w^-} and that either v^+ or v^- belongs to $sp(s,g)$ [6, 7, 8].

The algorithms we propose all follow the same high level strategy as described by Klein [6]. The general idea is that the robot moves from one point that is known to lie on $sp(s,g)$ to a point on $sp(s,g)$ that is closer to g by a sequence of moves as described below.

Algorithm High Level Strategy

Input: a street (P, s, g) and a path $\mathcal{P}_{sr}$ from s to the current position of the robot r;

while v^+ and v^- are defined **and** g is not reached **do**

Compute a path $\mathcal{P}_{rt}$ from r to some point t on $\overline{v^+v^-}$;

Follow the path $\mathcal{P}_{rt}$ until one of the following events occurs:

a) g becomes visible:
  the robot moves directly to g;

b) P_{w^+} or P_{w^-} becomes visible:
  if P_{w^+} is visible
    then the robot moves to v^-;
    else the robot moves to v^+;

c) v^+, v^-, and the current robot position r become collinear (see Figure 2b):
  the robot moves to the closer of v^+ and v^-;

d) v^+ or v^- changes;

Let r be the current robot position;

Compute $V(r)$ and v^+ and v^- anew;

end while;

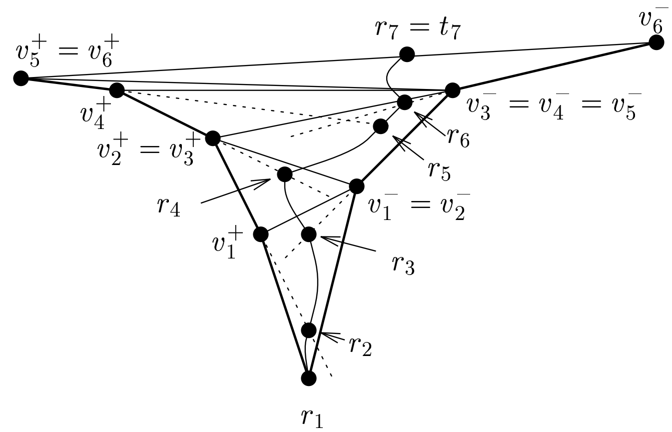

Figure 3: An example of the execution of the *High Level Strategy*.

An example of how the robot moves is given in Figure 3. The only detail left open by the above description is what path $\mathcal{P}_{rt}$ to choose which is called a "low-level strategy" [6]. In the following we investigate two low-level strategies and analyse their performance.

3 *lad* and Beyond

In this section we consider a new strategy which is similar in spirit to the first strategy that was proposed to traverse streets [6]. In [6] the strategy *lad* is presented which is based on the idea of minimizing the *l*ocal *a*bsolute *d*etour. The importance of *lad*—apart from being the first strategy proposed—lies in the fact that it is the only strategy that uses a heuristic optimality criterion to guide the robot. All other strategies that have been presented have no comparable feature. The well-chosen heuristic and its excellent performance in practice make *lad* a very attractive strategy. Unfortunately, it seems that it is exactly this property that makes *lad* also extremely difficult to analyse. As mentioned before the best performance guarantee is $1+\pi/2+\sqrt{1+\pi^2/4}$ (~ 4.44) which seems to be a very loose bound considering that the competitive ratio of the strategy observed in practice is less than 1.8 [6].

In the following we present a slight variant of *lad* which we call *continuous lad* that also follows the paradigm of minimizing the local absolute detour but whose analysis turns out to be much simpler and tighter. It can be shown to achieve a competive ratio of ~ 2.03 which is slighly better than the best performance guarantee of ~ 2.05 known so far [8]. We start out with some additional definitions and observations.

3.1 Preliminary Results for Low-Level Strategies

A first observation we can make about the high level strategy is that if one of the Cases a)–c) occurs, then we know which of v^+ or v^- belongs to $sp(s,g)$ and, hence, the competitive ratio of the strategy is given by ratio of the length of the path that the robot travels between two points p and q which are on $sp(s,g)$ and the shortest path $sp(p,q)$ from p to q.

So in the following we assume that the robot starts out at a point $r_1 \in sp(s,g)$ and encounters a number of events of Category d). Each of these events correspond to one point r_i, $i \geq 2$, at which new left and right extreme entrance points v_i^+ and v_i^- appear and a new path $\mathcal{P}_i$ from r_i to a point t_i on $\overline{v_i^+v_i^-}$ is computed. Let d_i^+ be the distance of r_i to v_i^+ and d_i^- be the distance from r_i to v_i^-.

Given r_i, the point r_{i+1} is defined as the first point on $\mathcal{P}_i$ such that either the left or the right extreme entrance point of $V(r_{i+1})$ is different from v_i^+ or v_i^-, respectively (see Figure 4). At the point r_{i+1} the robot computes a new target point and a new path $\mathcal{P}_{i+1}$.

We denote the angle $\angle r_i v_i^+ r_{i+1}$ by α_i^+ and the angle $\angle r_{i+1} v_i^- r_i$ by α_i^-. The angle of $\angle v_i^+ r_i v_i^-$ is denoted by γ_i. We can make the following elementary observation about the angles γ_{i+1} and γ_i.

Observation 3.1 $\gamma_{i+1} = \gamma_i + \alpha_i^+ + \alpha_i^-$.

Let $a_i^+ = d_i^+ - (d_{i+1}^+ - d(v_i^+, v_{i+1}^+))$ and $a_i^- = d_i^- - (d_{i+1}^- - d(v_i^-, v_{i+1}^-))$. Note that either $d(v_i^+, v_{i+1}^+) = 0$ or $d(v_i^-, v_{i+1}^-) = 0$. Furthermore, note that the distance of r_{i+1} to v_i^+ is $d_i^+ - a_i^+$ and the distance of r_{i+1} to v_i^- is $d_i^- - a_i^-$. Let $\mathcal{V}_i^+$ be the shortest path

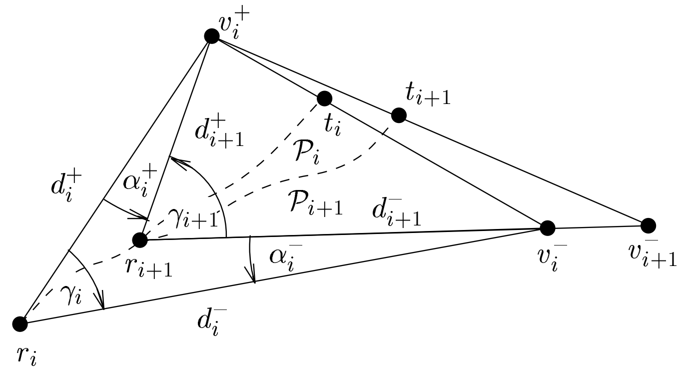

Figure 4: r_{i+1} is the first point on $\mathcal{P}_i$ where the left or right extreme entrance point changes.

from r_0 to v_i^+, and $\mathcal{V}_i^-$ the the shortest path from r_0 to v_i^-.

If the distance to v_i^+ and v_i^- decreases monotonously as the robot travels on $\mathcal{P}_i$, for all $0 \leq i \leq k$, then the length of $\mathcal{V}_i^+$ or $\mathcal{V}_i^-$ can be expressed by as d_i^+ plus the sum of the a_i^+ or d_i^- plus the sum of the a_i^-, respectively.

Lemma 3.2 *If, for all* $1 \leq i \leq k$, $d(r_{i+1}, v_i^+) \leq d_i^+$ *and* $d(r_{i+1}, v_i^-) \leq d_i^-$, *then we obtain, with the above definitions,*

$$\lambda(\mathcal{V}_i^+) = \sum_{j=0}^{i-1} a_j^+ + d_i^+ \quad \textit{and} \quad \lambda(\mathcal{V}_i^-) = \sum_{j=0}^{i-1} a_j^- + d_i^- .$$

3.2 The Strategy *lad*

We give a short description of the rationale behind *lad* as well as its definition, so as to stress both the differences and similarities between it and *continuous lad*.

If the robot has not been able to decide whether v_i^+ or v_i^- belongs to the shortest path from s to g after i steps, it chooses a new target point t_i on $\overline{v_i^+v_i^-}$ and the line segment $\mathcal{P}_i = \overline{r_i t_i}$ to travel from its current position r_i to t_i. Let $\mathcal{Q}_i$ be the path of the robot from r_1 to r_i and recall that $\mathcal{V}_i^+$ is the shortest path from r_0 to v_i^+ and $\mathcal{V}_i^-$ the shortest path from r_0 to v_i^-. If v_i^+ lies on the shortest path from s to g, then the local absolute detour is given by the distance the robot travels from r_1 to v_i^+ which is $\lambda(\mathcal{Q}_i)+\lambda(\mathcal{P}_i)+d(t_i, v_i^+)$ minus the length of the shortest path $\lambda(\mathcal{V}_i^+)$ from r_1 to v_i^+. A similar statement holds if v_i^- belongs to $sp(s,g)$. Hence, the maximum local absolute detour is minimized if

$$\begin{aligned} \lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i) + d(t_i, v_i^+) - \lambda(\mathcal{V}_i^+) &= \\ \lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i) + d(t_i, v_i^-) - \lambda(\mathcal{V}_i^-) & \qquad (1) \end{aligned}$$

and the point t_i on $\overline{v_i^+ v_i^-}$ is given by

$$d(v_i^+, t_i) \;=\; \frac{\lambda(\mathcal{V}_i^+) - \lambda(\mathcal{V}_i^-) + d(v_i^+, v_i^-)}{2}. \quad (2)$$

Note that $\lambda(\mathcal{V}_i^+) = \sum_{j=0}^{i-1} d(v_{j+1}^+, v_j^+)$ and $\lambda(\mathcal{V}_i^-) = \sum_{j=0}^{i-1} d(v_{j+1}^-, v_j^-)$ where we define $v_0^+ = v_0^- = r_1$.

3.3 The Strategy *continuous lad*

In the strategy *continuous lad* the robot also follows a path from r_i to t_i where t_i is determined by Equation 2; however, the robot does not move on a straight line segment. Instead, it moves on a path $\mathcal{P}_i$ such that for *each point* r on $\mathcal{P}_i$ the local absolute detour is minimized. Instead of being a line segment, $\mathcal{P}_i$ is now part of a hyperbola. Although this slight modification may seem to complicate the analysis further, it, in fact, allows to prove a much tighter upper bound on the competitive ratio for *continuous lad* than for *lad*. Note also that although the strategies seem almost identical, the points r_i at which the left or right extreme entrance points change for *lad* and *continuous lad* can be quite far apart.

We assume that the robot travels along a path $\mathcal{P}_i$ from r_i to r_{i+1} such that every point r on $\mathcal{P}_i$ satisfies Equation 1 if we replace t_i by r and $\mathcal{P}_i$ by $\mathcal{P}_i(r_i, r)$. If the robot follows the strategy *continuous lad*, then $a_i^+ = a_i^-$ and the location of t_i is only determined by d_i^+ and d_i^-.

Lemma 3.3 *If the robot travels on a path $\mathcal{P}_i$ such that for all $r \in \mathcal{P}_i$,*

$$\lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i(r_i, r)) + d(r, v_i^+) - \lambda(\mathcal{V}_i^+) \;=\; \lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i(r_i, r)) + d(r, v_i^-) - \lambda(\mathcal{V}_i^-),$$

then $a_i^+ = a_i^- > 0$.

Proof: The proof is by induction on i. For $i = 1$, we have $\lambda(\mathcal{V}_1^+) = d(r_1, v_1^+)$ and $\lambda(\mathcal{V}_1^-) = d(r_1, v_1^-)$ and if we set $r = r_2$, then the above equation immediately yields

$$a_1^- = d(r_1, v_1^-) - d(r_2, v_1^-) = d(r_1, v_1^+) - d(r_2, v_1^+) = a_1^+.$$

Since the robot moves into the interior of the triangle (r_1, v_1^+, v_1^-) it is easy to see that $a_1^+ > 0$. So now assume the claim is true, for all $1 \le i \le k-1$. Since $d(r_{i+1}, v_i^+) = d_i^+ - a_i^+ < d_i^+$, for all $1 \le i \le k-1$, Lemma 3.2 holds and $\lambda(\mathcal{V}_k^+) = \sum_{j=0}^{k-1} a_j^+ + d_k^+$. Similarly, we have $\lambda(\mathcal{V}_k^-) = \sum_{j=0}^{k-1} a_j^- + d_k^-$. By the induction hypothesis $\sum_{j=0}^{k-1} a_j^+ = \sum_{j=0}^{k-1} a_j^-$ and the above equation again yields

$$a_k^- = d_k^- - d(r_{k+1}, v_k^-) = d_k^+ - d(r_{k+1}, v_k^+) = a_k^+.$$

$a_k^+ > 0$ can be seen as in the case $i = 1$. □

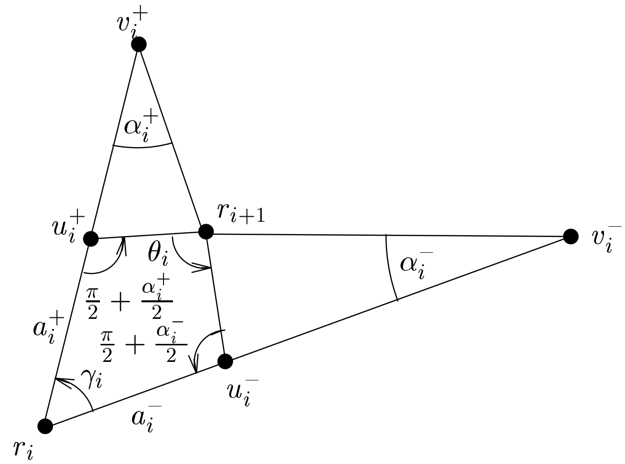

Figure 5: Illustrating the proof of Lemma 3.4.

3.4 Analysis of the Strategy *continuous lad*

In the following we assume that the robot travels on a path $\mathcal{P}_i$ such that, for all points r_{i+1} on $\mathcal{P}_i$ the distances a_i^+ and a_i^- are the same, i.e. $a_i^+ = a_i^- = a_i$. We analyse a step of the High-Level-Algoritm which consists of k consecutive events of Category d) and one event in the Categories a)–c). As a first step we compute an upper bound on the length of the path that is given by the line segments connecting the points r_i to r_{i+1}. In a second step we then show how to extend this analysis to $\mathcal{Q}_k$.

We present two bounds for the length of the path connecting the points r_j, $1 \le j \le k$. The first bound gives a good approximation if the angle γ_i is small and the second bound approximates large angles.

Lemma 3.4 *If $a_i^+ = a_i^-$, then $d(r_i, r_{i+1}) \le a_i^+ / \cos(\gamma_{i+1}/2)$.*

Proof: Let r_{i+1} be chosen such that $a_i^+ = a_i^-$. Consider the quadrilateral formed by r_i, u_i^+, u_i^-, and r_{i+1} as shown in Figure 5.

The location of r_{i+1} is completely determined by the angles α_i^+, α_i^-, and γ_i. The angle of the quadrilateral formed at u_i^+ is $(\pi + \alpha_i^+)/2$ and at u_i^- it is $(\pi + \alpha_i^-)/2$. Since $\alpha_i^+ + \alpha_i^- + \gamma_i = \gamma_{i+1}$, we can choose α_i^+ and α_i^- in order to maximize the distance of r_{i+1} to r_i. Let $\theta_i = \angle u_i^- r_{i+1} u_i^+$. Note that $\theta_i = 2\pi - \gamma_i - (\pi + \alpha_i^+)/2 - (\pi + \alpha_i^-)/2 = \pi - \gamma_i - \alpha_i^+/2 - \alpha_i^-/2$.

Let $\delta_1 = \angle u_i^- u_i^+ r_{i+1}$ and $\delta_2 = \angle r_{i+1} u_i^- u_i^+$. Hence, $\delta_1 + \delta_2 = \pi - \theta_i$, where θ_i is fixed. Furthermore, we introduce a coordinate system such that the origin is located at u_i^+, $u_i^- = (1, 0)$, and r_i is located on the line $L = \{(x, y) \mid x = 1/2\}$. Let C be the circle that passes through u_i^-, r_{i+1}, and u_i^+ with center c. The

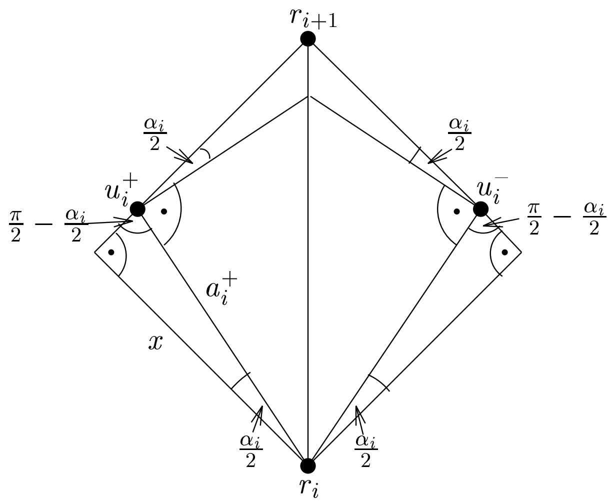

Figure 6: Choosing γ_i.

path of all points with $\delta_1 + \delta_2 = \pi - \theta_i$ is the arc A of C from u_i^- to u_i^+ that contains r_{i+1} (see [11, Sec. 16, Th. 4]).

We claim that $d(r_i, r_{i+1})$ is maximal for $\delta_1 = \delta_2$. Let p be the topmost point of the arc A, i.e., $\delta_1 = \delta_2$ if $r_{i+1} = p$. We note that c is located on the line L. If c is above r_i, then the circle with center r_i and radius $d(r_i, p)$ contains C and, hence, p is the point with maximal distance to r_i.

Let q be the point $(1/2, 0)$. We claim that c is above r_i. In order to show this we compute $d(r_i, q)$ and $d(c, q)$. The angle $\angle q r_i u_i^+$ is obviously $\gamma_i/2$. Hence, $d(r_i, q) = 1/2 \cot(\gamma_i/2)$. By [11, Sec. 16, Th. 2] the angle $\angle u_i^- c u_i^+$ equals $2\pi - 2\theta_i = 2\gamma_i + \alpha_i^+ + \alpha_i^-$ and $d(c, q) = 1/2 \cot(\pi - \theta_i) = 1/2 \cot(\gamma_i + \alpha_i^+/2 + \alpha_i^-/2) < 1/2 \cot(\gamma_i/2) = d(r_i, q)$ as claimed.

Therefore, we can assume $\alpha_i^+ = \alpha_i^-$ and we have the configuration displayed in Figure 6. Since

$$\cos\left(\frac{\alpha_i^+}{2}\right) = \frac{x}{a_i^+} \text{ and } \cos\left(\frac{\alpha_i^+ + \gamma_i}{2}\right) = \frac{x}{d(r_i, r_{i+1})},$$

we obtain

$$d(r_i, r_{i+1}) = \frac{\cos(\alpha_i^+/2) a_i^+}{\cos((\gamma_i + \alpha_i^+)/2)}.$$

With $(\alpha_i^+ + \gamma_i)/2 \le \gamma_{i+1}/2 < \pi/2$ and $\cos(\alpha_i^+/2) \le 1$ the claim follows. □

For large angles we make use of the following observation.

Lemma 3.5 *If* $a_i^+ = a_i^-$ *and the angle at the robot position is* γ_i*, then*

$$d(r_i, r_{i+1}) \le \min\{\alpha_i^+ d_i^+, \alpha_i^- d_i^-\} + a_i.$$

We now can analyse the competitive ratio of *continuous lad*. To do so, we first consider the path $\mathcal{P}_i'$ that consists of the line segments connecting the points r_i. For the analysis we split the execution of the strategy into two parts. In the first part we consider the length of the path of the robot until the angle between v_i^+ and v_i^- is equal to some angle γ and in the second part we consider the length of the remaining path of the robot. In order to see that it is possible to chose γ to be any value that is greater than or equal to γ_0, we argue that $\angle v_i^+ r_i(t) v_i^-$ is a continuous and monotonously increasing function as the robot travels on $\mathcal{P}_i$.

We assume that the robot has reached the point r_i and the angle $\angle v_i^+ r_i v_i^-$ is γ_i. At r_i the robot chooses a point t_i on the line segment $\overline{v_i^+ v_i^-}$ and moves on the path $\mathcal{P}_i$ from r_i to t_i until a new left or right extreme entrance point becomes visible. Let $r_i(t)$ denote the position of the robot at time t while traveling on $\mathcal{P}_i$. We assume that $r_i(0) = r_i$ and $r_i(1) = t_i$. We denote the angle $\angle v_i^+ r_i(t) v_i^-$ by $\gamma_i(t)$. Since $d(r_i(t), v_i^+)$ and $d(r_i(t), v_i^-)$ both decrease monotonically and continuously with t, the angles $\alpha_i^+(t) = \angle r_i v_i^+ r_i(t)$ and $\alpha_i^-(t) = \angle r_i v_i^- r_i(t)$ are continuous and monotonously increasing functions of t and, therefore, $\gamma_i(t) = \gamma_i + \alpha_i^+(t) + \alpha_i^-(t)$ is also a continuous and monotonously increasing function of t. Hence, if $\gamma_0 \le \gamma \le \gamma_k$, there is one $0 \le i_0 \le k$ and one $0 \le t_0 \le 1$ with $\gamma_{i_0}(t_0) = \gamma$. We can assume in the following that the robot executes Strategy *continuous lad* until either of the extreme entrance points is undefined or $\gamma_{i_0}(t_0) = \gamma$, for some i_0 and t_0. We insert the point $r_{i_0+1} = r_{i_0}(t_0)$ into the sequence of points $(r_1, \ldots, r_k)$ so that now there are $k+1$ points r_i. If $\gamma_0 \ge \gamma$, then we define $r_{i_0+1} = r_0$. In both cases we have $\gamma_{i_0+1} \ge \gamma$.

In order to analyse the competitive ratio of *continuous lad* we need to estimate the length of the path $\mathcal{P}_i$ that is the concatenation of parts of hyperbolas. Recall that if $\Lambda(\mathcal{P}_i)$ is the set of all finite sequences of points on $\mathcal{P}_i$ that occur in order, i.e. $\Lambda(\mathcal{P}_i) = \{(q_1, \ldots, q_m) \mid m \ge 1,\ q_j \in \mathcal{P}_i$, for all $1 \le j \le m$, and q_{j+1} occurs after q_j on $\mathcal{P}_i\}$, then $\lambda(\mathcal{P}_i) = \sup\{\sum_{j=1}^{m-1} d(q_j, q_{j+1}) \mid (q_1, \ldots, q_m) \in \Lambda(\mathcal{P}_i)\}$. We will make use of this definition, to estimate the length of $\mathcal{P}_i$.

So consider a sequence $(q_1, \ldots, q_m) \in \Lambda(\mathcal{P}_i)$. Since we are interested in obtaining a supremum and adding points only increases $\sum_{j=1}^{m-1} d(q_j, q_{j+1})$, we can assume that $(r_1, \ldots, r_k)$ is a subsequence of $(q_1, \ldots, q_m)$. We define as before v_j^+ to be the left entrance point of $V(q_j)$, $d_j^+ = d(q_j, v_j^+)$, and

$a_j^+ = d_j^+ - (d_{j+1}^+ - d(v_i^+, v_{i+1}^+))$. $v_j^-, d_j^-, a_j^-, \gamma_j$, etc. are defined analogously. Note that with the above definitions Lemmas 3.2 and 3.3 still hold which implies that Lemmas 3.4 and 3.5 hold as well.

So let $\mathcal{R}_m$ be the path connecting the points $q_1, \ldots, q_m$ by line segments. We now observe that if the angle γ_j at q_j is less than or equal to γ, then Lemma 3.4 yields that $d(q_{j-1}, q_j) \leq a_{j-1}/\cos(\gamma/2)$. Otherwise, Lemma 3.5 yields that $d(q_j, q_{j+1}) \leq \alpha_j^+ d_j^+ + a_j^+$ and $d(q_j, q_{j+1}) \leq \alpha_j^- d_j^- + a_j^-$. If j_0 is the index, such that $\gamma_{j_0} = \gamma$, then we obtain the following analysis of the length of $\mathcal{R}_m$ where we assume w.l.o.g. that $v_{m-1}^+ \in sp(s, g)$.

$$\begin{aligned}
&\sum_{j=1}^{m-1} d(q_j, q_{j+1}) + d_m^+ \\
&= \sum_{j=1}^{j_0} d(q_j, q_{j+1}) + \sum_{j=j_0+1}^{m-1} d(q_j, q_{j+1}) + d_m^+ \\
&\leq \sum_{j=1}^{j_0} \frac{a_j^+}{\cos(\gamma/2)} + \sum_{j=j_0+1}^{m-1} (\alpha_j^+ d_j^+ + a_j^+) + d_m^+ \\
&\leq \sum_{j=1}^{j_0} \frac{a_j^+}{\cos(\gamma/2)} + \left(\lambda(\mathcal{V}_m^+) - \sum_{j=1}^{j_0} a_j^+\right) \sum_{j=j_0+1}^{m-1} \alpha_j^+ + \\
&\quad \sum_{j=j_0+1}^{m-1} a_j^+ + d_m^+ \\
&\leq \frac{1}{\cos(\gamma/2)} \sum_{j=1}^{j_0} a_j^+ + (\pi - \gamma + 1)\left(\lambda(\mathcal{V}_m^+) - \sum_{j=1}^{j_0} a_j^+\right) \\
&\leq \max\{1/\cos(\gamma/2), \pi - \gamma + 1\}\lambda(\mathcal{V}_m^+)
\end{aligned}$$

So let the angle γ be chosen such that the maximum of $\{1/\cos(\gamma/2), \pi - \gamma + 1\}$ is minimized, i.e., that $1/\cos(\gamma/2) = \pi - \gamma + 1$. By numerical evaluation we obtain that $\gamma \sim 2.111$. Hence, $(\pi - 2.111) + 1$ (~ 2.03) is an upper bound on the length of the path $\mathcal{R}_m$ that connects the points q_j by straight line segments. Since $(q_1, \ldots, q_m)$ is chosen arbitrarily from $\Lambda(\mathcal{P}_i)$, the supremum of $\{\sum_{j=1}^{m-1} d(q_j, q_{j+1}) \mid (q_1, \ldots, q_m) \in \Lambda(\mathcal{P}_i)\}$ is also bounded by 2.03.

4 Changing the Strategy

If we take a closer look at the analysis of Strategy *continuous lad*, then we notice that the ratio obtained for small angles is much tighter than the bound on large angles. Unfortunately, it is not obvious how to improve the analysis. However, there is another option. Since the robot can measure the angle between v_i^+ and v_i^- at its position, it is possible to change the strategy once a certain threshold is reached. We assume that the robot has encountered k events of category d) in *continuous lad* and switches to a new strategy at point r_k.

In the following we consider the Strategy *Move-in-Quadrant* which was already presented in [8] but we provide a tighter analysis if the angle γ_k is larger than $\pi/2$.

In order to present the strategy we need the notion of a projection of a point. The *orthogonal projection* p' of a point p onto a line segment l is defined as the point of l that is closest to p.

Strategy Move-in-Quadrant
Input: A point r_k in P such that the angle
  $\gamma_k = \angle r5r[k] r_k v_k^+$ is $\geq \pi/2$;
 $i := k$;
 while v_i^+ and v_i^- of $V(r_i)$ are defined **do**
(1) Move to the orthogonal projection r_{i+1} of r_k onto the line segment l_i from v_i^+ to v_i^-;
  Compute the points v_{i+1}^+ and v_{i+1}^- of the visibility polygon $V(r_{i+1})$ of r_{i+1};
  $i := i + 1$;
 end while;

The correctness of the strategy has been proven in [8]. Note that the Strategy *Move-in-Quadrant* also follows the schema of the High-Level-Strategy except that events of Category d) are replaced by events of Category d′): $t_i = r_{i+1}$ is reached.

4.1 Analysis of the Strategy *Move-in-Quadrant*

In the following we assume that the Strategy *Move-in-Quadrant* has stopped after $m - k$ iterations. As before the shortest path goes either through v_i^+ or v_i^- [8].

Recall that γ_k is defined to be the angle $\angle v_k^+ r_k v_k^-$ which we assume to be greater than or equal to $\pi/2$. We introduce a coordinate system where r_k is the origin and the angle δ_k^- between the x-axis and the line segment $\overline{r_k v_k^-}$ equals the angle δ_k^+ between the y-axis and the line segment $\overline{r_k v_k^+}$. We define $\delta_k = \delta_k^+ = \delta_k^-$.

Now suppose that we have arrived at point r_i and move to point r_{i+1} in the next iteration. To simplify the analysis, we consider the line segment l_i' from the intersection point of $\overline{v_i^+ v_i^-}$ with the line through r_k and v_k^+ to the intersection point of $\overline{v_{i+1}^+ v_{i+1}^-}$ with the line through r_k and v_k^- as shown Figure 7.

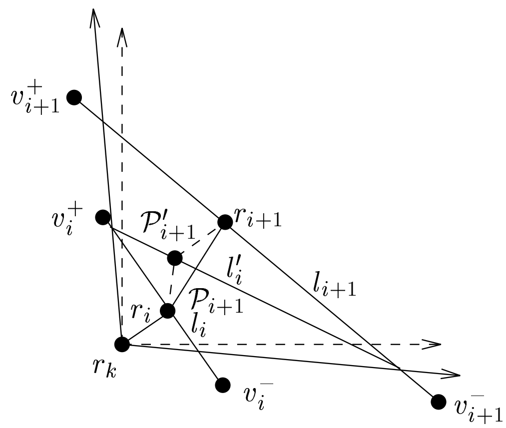

Figure 7: Introducing a new segment between l_i and l_{i+1}.

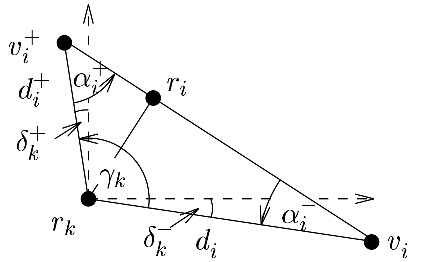

Figure 8: The location of r_i with respect to r_k.

The line segment l'_i is located between l_i and l_{i+1}. If we consider the path $\mathcal{P}'_i$ from r_k to r_i that visits the orthogonal projections of r_k onto the line segments l_j and l'_j in order, for $k \leq j \leq i$, then the length of $\mathcal{P}'_i$ is obviously greater than or equal to the length of $\mathcal{P}_i$. Furthermore, $\mathcal{P}_i$ and $\mathcal{P}'_i$ share the same start and end point. Hence, for the simplicity of exposition we assume in the following that v_i^+ and v_i^- are located on the line from r_k to v_k^+ and v_k^-, respectively, and that either $v_i^+ = v_{i+1}^+$ or $v_i^- = v_{i+1}^-$.

Let L_i be the length of the path $\mathcal{P}_i$ traveled by the robot from r_k to reach r_i; let α_i^- be the angle $\angle r_k v_i^- r_i$, and d_i^- the distance $d(r_k, v_i^-)$ (see Figure 8). Similarly, let α_i^+ be the angle $\angle r_i v_i^+ r_k$ and d_i^+ the distance $d(r_k, v_i^+)$. We define the angle α_i as $\min\{\pi/2 - \alpha_i^+, \pi/2 - \alpha_i^-\}$ and the distance d_i as $\min\{d_i^+, d_i^-\}$. Note that $\pi/2 - \alpha_i^+ + \pi/2 - \alpha_i^- = \gamma_k$ and, therefore, $\alpha_i^+ + \alpha_i^- + 2\delta_k = \pi/2$ or $\pi/2 - \alpha_i^+ = \alpha_i^- + 2\delta_k$ and $\pi/2 - \alpha_i^- = \alpha_i^+ + 2\delta_k$. In particular, $\alpha_i = \pi/2 - \alpha_i^+$ if and only if $d_i = d_i^+$.

Our approach to analyze our strategy is based on the idea of a potential function Q_i [8]. It is our aim to show that $L_i + Q_i \leq (\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2})d_i$, for all $k \leq i \leq k$, where we define $Q_i = \alpha_i d_i$. So suppose the robot has reached the point r_i and $L_i \leq (\gamma_k/2 + \cot\gamma_k/2 - \alpha_i)d_i$ and d_i is equal to the distance between r_0 and v_i^-. For simplicity of description we assume that the distance from r_k to v_i^+ is 1.

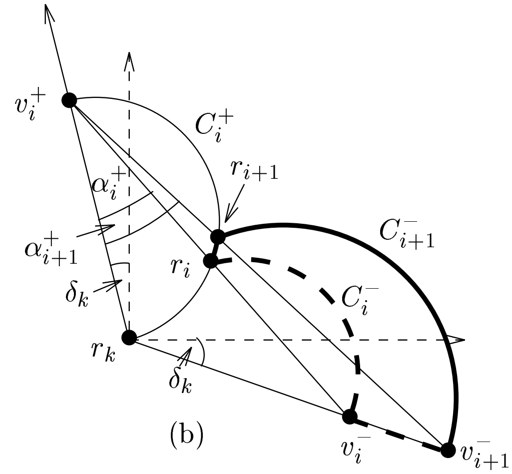

Figure 9: Case 2 if the robot moves from r_i to r_{i+1}.

For the distance d_i^- we obtain

$$d_i^- = \frac{\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)}. \tag{3}$$

The robot moves now from r_i to r_{i+1}. We distinguish two cases.

Case 1 $\alpha_i^+ > \alpha_{i+1}^+$ or $\alpha_{i+1}^+ > \alpha_{i+1}^-$.
These two Cases can be analysed exactly as the Cases 1 and 3 the analysis of *Move-in-Quadrant* of [8].

Case 2 $\alpha_i^+ \leq \alpha_{i+1}^+ \leq \alpha_{i+1}^-$ (see Figure 9).
Hence, $d_{i+1} = \sin\alpha_{i+1}^+/\cos(\alpha_{i+1}^+ + 2\delta_k)$. Note that Note that r_{i+1} is on the circle C_i^+ with center at $c_i^+ = 1/2(-\sin\delta_k, \cos\delta_k)$ and radius 1/2. The arc a_i^+ of C_i^+ from r_i to r_{i+1} has length $2(\pi/2 - \alpha_i^+ - (\pi/2 - \alpha_{i+1}^+))1/2$. Clearly, the line segment $\overline{r_i r_{i+1}}$ is shorter than the arc a_i^+. Hence,

$$\begin{aligned} L_{i+1} &= L_i + d(r_i, r_{i+1}) \\ &\leq \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2}\right)d_i - \alpha_i d_i + \alpha_{i+1}^+ - \alpha_i^+ \end{aligned}$$

We want to show that

$$\left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \alpha_i\right)d_i + (\alpha_{i+1}^+ - \alpha_i^+) \leq \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \alpha_{i+1}\right)d_{i+1} \tag{4}$$

or

$$\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} \geq \frac{\alpha_{i+1}d_{i+1} - \alpha_i d_i + \alpha_{i+1} - \alpha_i}{d_{i+1} - d_i}$$

with $\gamma_k - \pi/2 \le \alpha_i \le \alpha_{i+1} \le \pi/4 - \delta_k$. If define $\beta_i = \alpha_{i+1} - \alpha_i = \alpha_{i+1}^+ - \alpha_i^+$ and

$$f(\alpha_i, \beta_i, \delta_k) = \frac{\beta_i + \frac{(\alpha_i^+ + \beta_i + 2\delta_k)\sin(\alpha_i^+ + \beta_i)}{\cos(\alpha_i^+ + \beta_i + 2\delta_k)} - \frac{(\alpha_i^+ + 2\delta_k)\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)}}{\frac{\sin(\alpha_i^+ + \beta_i)}{\cos(\alpha_i^+ + \beta_i + 2\delta_k)} - \frac{\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)}},$$

then we want to prove that $f(\alpha, \beta, \delta) \le \gamma/2 + \cot\gamma/2$, where $\gamma = \pi/2 + 2\delta$, for all $(\alpha, \beta, \delta) \in \Delta = \{(x, y, z) \mid x \ge 0, y \ge 0, z \ge 0, x + y + z \le \pi/4\}$ since we assume that $\alpha_{i+1}^+ \le \alpha_{i+1}^-$, i.e. $\alpha_{i+1}^+ \le \pi/2 - \alpha_{i+1}^+ - 2\delta_k$ or $\alpha_i^+ + \beta_i + \delta_k \le \pi/4$.

By considering the partial derivatives of f w.r.t. α it can be easily shown that f is monotone w.r.t. α and, therefore,

$$\max f(\alpha, \beta, \delta) \quad = \quad \max f\left(\frac{\pi}{4} - \beta - \delta, \beta, \delta\right)$$

If we define $g(\beta, \delta) = f\left(\frac{\pi}{4} - \beta - \delta, \beta, \delta\right)$, then

$$\frac{\partial}{\partial\beta}\left(\frac{\partial g}{\partial\beta}(\beta, \delta)\frac{1}{2}\frac{1}{\cos 2\delta(\cos 2\beta - 1)}\right) = \frac{1}{2}\sin(4\delta)\cos(2\beta) - \cos 2\delta\cos 2\beta - \frac{1}{2}\sin 4\delta + \cos 2\delta$$

which is equal to 0 if and only if $\cos(2\beta) = 1$ or $\sin(4\delta)/2 = \cos 2\delta$, the latter of which only holds for $\delta = \pi/4$. Furthermore, since $0 \le \beta \le \pi/4$, the former holds only if $\beta = 0$. Therefore, we can see easily that $\frac{\partial}{\partial\beta}\left(\frac{\partial g}{\partial\beta}(\beta, \delta)\frac{1}{2}\frac{1}{\cos 2\delta(\cos 2\beta - 1)}\right) \le 0$ and, thus, $\frac{\partial g}{\partial\beta}(\beta, \delta)/2(\cos 2\delta(\cos 2\beta - 1))$ is monotonously decreasing in β.

It can be easily checked that

$$\lim_{\beta\to 0}\frac{\partial g}{\partial\beta}(\beta, \delta) = 0$$

and, therefore, $\frac{\partial g}{\partial\beta}(\beta, \delta) \le 0$, for all $0 \le \beta \le \pi/4 - \delta$. This in turn implies that g is monotonously decreasing in β and

$$\begin{aligned}
&\max_{(\alpha,\beta,\delta)\in\Delta} f(\alpha, \beta, \delta) \\
&= \max_{\beta\in[0,\pi/4-\delta]} g(\beta, \delta) \\
&= \lim_{\beta\to 0}\frac{\pi}{4} + \delta + \frac{\beta}{2} + \frac{\beta\cos(\beta + 2\delta)}{2\sin\delta} + \frac{\beta\cos\beta}{\sin\beta\cos 2\delta} - \\
&\quad \frac{3\beta\sin(\beta + 2\delta)}{4\sin\beta\cos 2\delta} + \frac{\beta\sin(\beta - 2\delta)}{4\sin\beta\cos 2\delta} - \frac{\beta\cos(\beta + 4\delta)}{4\sin\beta\cos 2\delta} \\
&= \frac{\pi}{4} + \delta + \cot\left(\frac{\pi}{4} + \delta\right) \\
&= \frac{\gamma}{2} + \cot\left(\frac{\gamma}{2}\right)
\end{aligned}$$

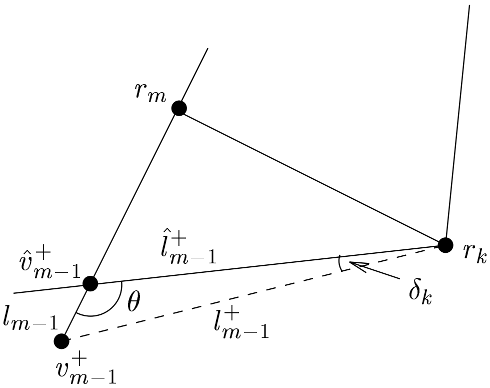

Figure 10: Bounding the final competitive ratio.

where $\gamma = \pi/2 + 2\delta$ as claimed.

In fact we have shown the following lemma.

Lemma 4.1 *For all $k \le i \le m$,*

$$\frac{\gamma_k}{2} + \cot\left(\frac{\gamma_k}{2}\right) \ge \max\left\{\frac{L_i + d(r_i, v_i^+)}{d(r_k, v_i^+)}, \frac{L_i + d(r_i, v_i^-)}{d(r_k, v_i^-)}\right\}$$

4.2 The Final Ratio

In order to obtain the final competitive ratio for one step we have to take into account that the robot has to move to either v_{m-1}^+ or v_{m-1}^-. If v_m^- is undefined, then v_{m-1}^+ belongs to the shortest path from s to g. Lemma 4.1 gives an upper bound on the maximum distance the robot travels in order to reach $\hat{v}_{m-1}^+$ in Figure 10 which is located on the line through r_k and v_k^+.

Let l_{m-1} be the line segment between v_{m-1}^+ and $\hat{v}_{m-1}^+$ and θ the angle between $\hat{l}_{m-1}^+$ and l_{m-1}. The length of l_{m-1}^+ grows monotonously with θ if the lengths of $\hat{l}_{m-1}^+$ of l_{m-1} are fixed. Hence, the maximum ratio of $(c\lambda(\hat{l}_{m-1}^+) + \lambda(l_{m-1}))/\lambda(l_{m-1}^+)$ is achieved for the minimum angle θ which is $\theta = \gamma_k = \pi/2 + 2\delta_k$. Let the length of $\hat{l}_{m-1}^+$ be d_1 and the length of l_{m-1} be d_2. Hence, the maximum distance traveled by the robot from r_k to v_{m-1}^+ is bounded by

$$F(\delta_k) = \max\frac{c(\delta_k)d_1 + d_2}{\sqrt{d_1^2 + d_2^2 - 2d_1d_2\cos(\frac{\pi}{2} + 2\delta_k)}}.$$

where $c(\delta) = \pi/4 + \delta + \cot(\pi/4 + \delta)$. This maximum is achieved at

$$d_2 = d_1\frac{1 - c(\delta_k)\sin 2\delta_k}{c(\delta_k) - \sin 2\delta_k}$$

and yields a value of

$$F(\delta_k) = \frac{c(\delta_k) + \frac{1-c(\delta_k)\sin 2\delta_k}{c(\delta_k)-\sin 2\delta_k}}{\sqrt{1+\left(\frac{1-c(\delta_k)\sin 2\delta_k}{c(\delta_k)-\sin 2\delta_k}\right)^2 + 2\frac{(1-c(\delta_k)\sin 2\delta_k)\sin 2\delta_k}{c(\delta_k)-\sin 2\delta_k}}}.$$

The same analysis applies if v_m^+ is undefined.

If we combine this with Lemma 3.4, we obtain the following upper bound on the distance traveled by the robot if the shortest path from s to g goes through v_{m-1}^+. Recall that $\mathcal{P}_i$ is the path the robot follows from point r_i to r_{i+1}, where we set $r_{m+1} = v_{m-1}^+$.

$$\begin{aligned}\sum_{j=0}^{m} \lambda(\mathcal{P}_i) = &= \sum_{j=0}^{k-1} \lambda(\mathcal{P}_i) + \sum_{j=k}^{m} d(r_j, r_{j+1}) \\ &\leq \frac{1}{\cos(\gamma_k/2)} \sum_{j=0}^{k} a_j^+ + F(\delta_k)\lambda(\mathcal{V}_{m-1}^+) \\ &\leq \max\left\{\frac{1}{\cos(\gamma_k/2)}, F(\delta_k)\right\} \lambda(\mathcal{V}_{m-1}^+)\end{aligned}$$

with $\gamma_k = \pi/2 + 2\delta_k$. Again the minimum competitive ratio is achieved if both the terms in the maximum are equal. This yields a value of 1.91 for γ_k and a competitive ratio of ~ 1.73.

5 Conclusions

We have presented two strategies for a robot to search in streets if it is given the visibility map of its local surroundings. The strategies proposed use the same "high level strategy" as outlined by Klein [6]. In the strategy *continuous lad* the resulting path followed by the robot is a concatenation of parts of hyperbolas. Though the path generated by the strategy is fairly complicated its analysis turns out to be much simpler than the analysis of the similar strategy *lad*.

The strategy proposed has a relatively good competitive ratio of 2.03. Surprisingly, this strategy, combined with the previously best known 2.05-competitive strategy results in a hybrid strategy with a competitive ratio of 1.73.

Often the idealistic assumption that a robot can follow a precomputed path without deviation is violated by real life robots. An interesting open problem is, therefore, if it is possible for a robot to traverse a scene with a predetermined maximal navigational error per unit traversed at a predetermined competitive ratio. Also the gap between the lower bound of $\sqrt{2}$ and the upper bound of 1.73 for search strategies in streets is still significant and needs to be improved.

References

[1] R. Baeza-Yates, J. Culberson and G. Rawlins. "Searching in the plane", *Information and Computation*, Vol. **106**, (1993), pp. 234-252.

[2] A. Blum, P. Raghavan and B. Schieber. "Navigating in unfamiliar geometric terrain", *Proc. of 23rd ACM Symp. on Theory of Computing*, (1991), pp. 494-504.

[3] K-F. Chan and T. W. Lam. "An on-line algorithm for navigating in an unknown environment", *International Journal of Computational Geometry & Applications*, Vol. **3**, (1993), pp. 227-244.

[4] X. Deng, T. Kameda and C. Papadimitriou. "How to learn an unknown environment I: The rectilinear case", *Technical Report CS-93-04*, Dept. of Comp. Sci., York University, 1993. Also as *Proc. 32nd IEEE Symp. on Foundations of Comp. Sci.*, (1991), pp. 298-303.

[5] Ch. Icking. Ph. D. Thesis, Fernuniversität Hagen, 1994.

[6] R. Klein. "Walking an unknown street with bounded detour", *Computational Geometry: Theory and Applications* **1**, (1992), pp. 325-351.

[7] J. Kleinberg. "On-line search in a simple polygon", *Proc. of 5th ACM-SIAM Symp. on Discrete Algorithms*, (1994), pp. 8-15.

[8] A. Lopez-Ortiz and S. Schuierer. "Going home through an unknown street", Proc. of 4th Workshop on Data Structures and Algorithms, 1995, to appear.

[9] A. Lopez-Ortiz and S. Schuierer. "Simple, Efficient and Robust Strategies to Traverse Streets", Proc. 7th Canad. Conf. on Computational Geometry, 1995, to appear.

[10] A. Mei and Y. Igarashi. "Efficient strategies for robot navigation in unknown environment" *Proc. of 21st Intl. Colloquium on Automata, Languages and Programming*, (1994).

[11] E. Moise. "Elementary Geometry from and Advanced Standpoint", 2nd ed., Addison-Wesley, 1973.

[12] C. H. Papadimitriou and M. Yannakakis. "Shortest paths without a map", *Theoretical Computer Science* **84**, (1991), pp. 127-150.

[13] D. D. Sleator and R. E. Tarjan. "Amortized efficiency of list update and paging rules", *Communications of the ACM* **28**, (1985), pp. 202-208.