Searching for the Centre of a Circle *

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1 Introduction

Consider the problem of an agent or robot searching for the centre of a circle. The robot has only a limited number of capabilities. It can detect whether it is inside the circle or not. It can mark the point which it occupies. It can move in a straight line, possibly towards a mark which it made previously. It can make 90 and 180 degree turns. It can move to the middle of a line segment determined by two marked points, if it is on the line segment. The searcher starts at the edge of the circle. The objective is for the searcher to move to the centre of the circle as quickly as possible.

This problem is inspired by the problem of finding a lost skier buried in an avalanche. The members of a ski party in the backcountry carry transceivers which are used for search and rescue in case of burial in an avalanche. A transceiver is a piece of equipment similar to a walkie-talkie that can either transmit or receive a radio signal. When a transceiver in receiver mode is within range of a transceiver in transmit mode, the receiver emits an audible signal. All transceivers are equipped with a volume knob. At the highest setting transceivers have a reception range of 10-35 meters.

The allowed operations are those that an individual trained skier could be expected to perform reasonably well. However we do not recommend that the methods proposed here be used for that purpose until extensive field testing has been done, by experts in the field of avalanche rescue.

Initially all skiers set their transceivers to the transmit position. In case of an avalanche, skiers who were not buried switch their transceivers to the receive position and start a search. The search can be divided into three phases: (1) the search for an audible signal, (2) the search for the approximate origin of the signal once first contact is made, and (3) the search for the precise location of the signal. In this paper, we study phase (2); see [3] for instructions for the other phases.

During an avalanche search, time is critical. Recent rescue statistics show that after five minutes under the snow chances of survival drop rapidly, with fewer than 30 percent surviving 45 minutes under the snow [1] . Thus locating the victim quickly may well be the difference between life and death. In this paper, we assume that the location time is proportional to the distance that was travelled. We also assume that the lines of equal strength signals from the transceivers are concentric circles as a reasonable first approximation. That is, the audible signal area for a given volume setting is a circle, with lower settings forming concentric circles of smaller radii. The formal problem is thus as follows:

Problem 1 *Let $C_1, \ldots, C_k$ be concentric circles with radii $r_1 > \ldots > r_k > 0$. Given a point p on the boundary of C_1, find the centre c of the circles, where the only allowed operation is to test whether we are inside C_i, for some $1 \leq i \leq k$. The number k is known a priori, where is the location and radii of the circles are not.*

There is a "traditional" approach to solving this problem, which is taught in skiing manuals (see for example [3]). It uses "markers" (for example a glove or skipole) that the searcher leaves at some visited places, to be able to compute midpoints between visited segments. The outline of the strategy is as follows (see [3] for details):

1. Find a direction from p that leads inside circle C. (Obtained naturally from phase (1) of the search.)
2. Move from p in this direction until the signal fades. We have thus found a chord of the circle. Leave a marker at this point.
3. Turn $180°$ and move back until the signal fades. Leave a marker at this point. We have now found a chord of the circle. [This step is unnecessary in the first iteration since we have exactly returned to p.]
4. Move to the midpoint of the chord and reduce the volume until the signal can just be heard.
5. Make a $90°$ turn and iterate the process. [The manual suggests to repeat 3-4 times to avoid error. In theory, after the second time we are at the centre of the circles.]

This strategy is very similar to one used in [2] to find an approximation to the centre of a nearly circular compact set in the plane. There, three moves to the midpoint

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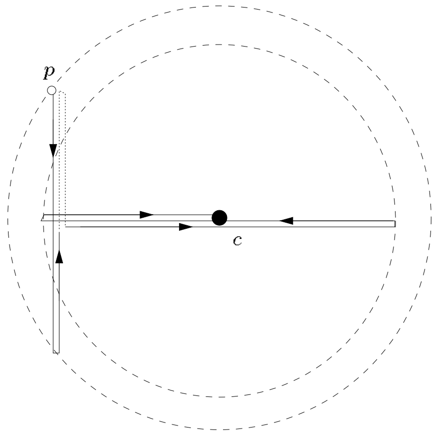

Figure 1: Handbook strategy for finding an avalanche victim.

of a chord are performed to find the approximate centre, because the first chord is not guaranteed to cross the diameter perpendicular to itself.

In [3] it is claimed that "Hundreds of practice sessions have shown the [...] technique to be the fastest way of finding a victim." In this paper, we show that at least in theory, faster ways are possible. We show an entirely different strategy (which we call the *right-angle strategy*) that does not use the volume knob, uses only two markers (as opposed to four markers of the traditional strategy), and consistently yields a shorter path to the centre. We also give other strategies that make more efficient use of the volume knob, or do not use markers at all.

We analyze our strategies under the competitive ratio framework. In the optimal strategy the searcher moves in a straight line from point p on the circle to the centre of the circle, and hence uses the radius r of the circle. The ratio of the distance traveled by the searcher compared to the radius of the circle gives the performance ratio.

Observe that in theory, we can have a competitive ratio arbitrarily close to 1, as illustrated in Figure 2. By inscribing a small circle centered at p, we can find two chords of C. The perpendicular bisectors of these two chords then is the centre point c. Notice that as the radius of the initial small circle chords approaches 0, the distance traversed approaches the optimal distance r.

However, in real-life settings we cannot assume a high enough precision to execute this strategy. We therefore considered strategies where the only geometric operations needed are angles of 90° and 180° and computation of midpoints. We also include error analysis with regards to errors in doing these operations.

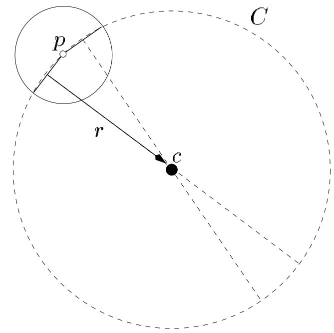

Figure 2: An optimal strategy.

2 Strategies without Volume

2.1 The traditional approach

We first analyze the traditional strategy. Note that in the worst-case the first chord may miss the inner circle, hence we may not be able to take advantage of the volume knob at all. We assume that we omit the part that returns us to point p.

Let θ be the angle formed by the perpendicular bisector of the first chord with the radius from point p. The length of travelled path may be either $3\,r\sin\theta - r\cos\theta + 4\,r$ or as much as $3\,r\sin\theta + r\cos\theta + 4\,r$, depending on whether the searcher turns towards or away from the centre. The competitive ratio is the larger, so it is

$$\mathcal{C} = \max_{\theta}\{3\,\sin\theta + \cos\theta + 4\},$$

By differentiating with respect θ we see that this expression is maximized when $\theta = \arctan(1/3)$, which gives a maximum of ≈ 7.16.

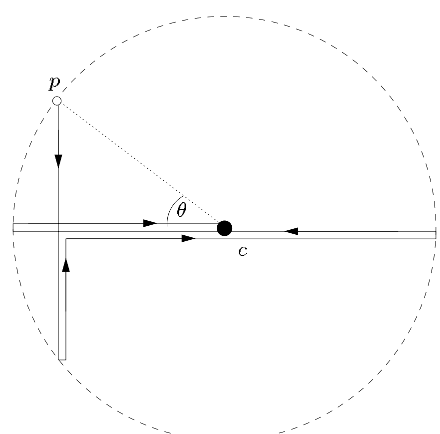

Figure 3: The traditional strategy.

We note here that the strategy could be improved slightly by noting in step 3 to which side the signal fades, and by then turning away from this direction in step 5, so that we reach the circle boundary along the shorter segment. The competitive ratio with this strategy then becomes at most 7.

2.2 The right-angle strategy

We now propose a different strategy for locating the centre point, which has less distance travelled and is (in our opinion) simpler than the traditional one. This strategy is illustrated in Figure 4 and proceeds as follows:

1. Leave a marker at p and determine a chord of the circle as before.
2. Turn $90°$ and move until the signal fades. If the signal fades right away turn $180°$ and move in the opposite direction until signal fades.
3. Mark the point where the signal fades.
4. Move towards the marker of Step 1, to the halfway point between the two markers.

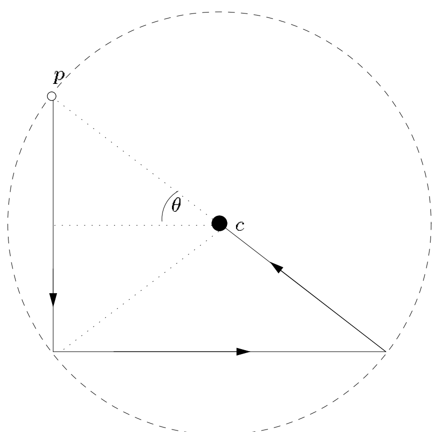

Figure 4: The right-angle strategy.

The total distance traversed is $2\,r\sin\theta + 2\,r\cos\theta + r$, which gives a competitive ratio of at most ≈ 3.82. Moreover, elementary math shows that this strategy outperforms the traditional strategy for all possible values of θ, regardless of the direction that the searcher turns.

Note also that the right-angle strategy requires only two markers and only one computation of midpoints, which makes it simpler to execute.

The strategy can be improved further if there are at least two searchers. (In this case, the time is assumed to be proportional to the maximum distance travelled among the two skiers.) If both searchers start at point p, traverse chords that are at 90°, and then move towards each other, they can reach the midpoint c such the searcher with the longer distance travels $\max\{2\cos\theta r, 2\sin\theta r\} + r$, leading to a competitive ratio of at most 3. One searcher can get to the centre in time required to travel at most ≈ 2.41 radii. This can be improved slightly further with more advanced strategies.

2.3 Error analysis

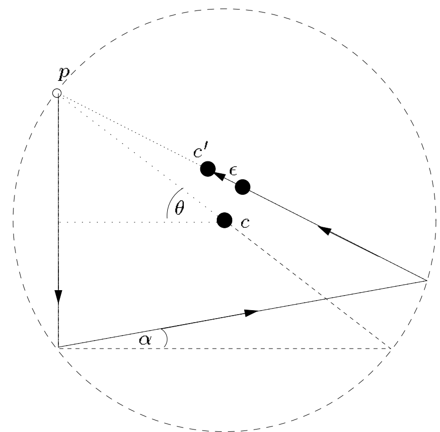

Figure 5: Error analysis for right-angle strategy.

In this section, we analyze the previous strategies with regards to how sensitive they are to errors. We consider two types of errors. First, when the searcher is making right angle turns, the new direction can differ from the desired direction by an angle up to α. Second, when the searcher is estimating the midpoint, the chosen point is in distance of up to ϵ from the real midpoint. We do not consider errors in $180°$ turns because (in the skiing case) the searcher has her own tracks available in this case, making the operation almost error-free. We also do not consider errors caused by moving in non-straight lines.

Possible errors for the right-angle strategy are illustrated in Figure 5. In step 2, the searcher can turn by angle $90° \pm \alpha$. This affects the position of the point located in step 3. The final location is further affected by an error of at most ϵ in determining the midpoint. The situation is similar in the case of the traditional strategy. However, in this case both the ϵ error in step 4 and the α error in step 5 affect the position of the segment needed in step 7. Locating the midpoint of the segment causes additional error of at most ϵ.

Table 1 shows the worst-case distances from the real centre for both traditional strategy and right-angle strategy. It is easy to see that while the right-angle strategy is more prone to angular errors, the traditional strategy has worse performance in mid-point estimation, in part because there are two mid-point estimations involved. It is also interesting that the error in the case of right-angle

	Traditional strategy	Right-angle strategy
Angular error only	$\cos\theta\sin\alpha$	$\sin\alpha$
Midpoint error only	$\sqrt{2}\epsilon$	ϵ
Both errors	$\sqrt{\cos^2\theta\sin^2\alpha+\epsilon\sin 2\alpha+\epsilon^2(1+\cos^2\alpha)}$	$\sqrt{\sin^2\alpha+\epsilon^2}$

Table 1: The maximum distance from the real centre.

strategy does not depend on angle θ.

Several simulations of both strategies were performed, using a normal distribution to sample both angular and mid-point estimation errors. In this experiment, the right-angle strategy performed slightly better than the traditional strategy (the average distance from the real centre was closer by about 10%).

3 Using Volume

Transceivers usually have different volume settings. Whenever we can lower the volume and still have an audible signal, we reduce the search space to a smaller circle. If we have a continuous volume knob, then we can find the centre very efficiently with a competitive ratio of $\sin\theta+\cos\theta\leq\sqrt{2}$. Namely, travel along the chord as before, but continuously lower the volume until we reach a local minimum. This is the midpoint of the chord. Continue at an angle of $90°$, in the direction where the volume can be decreased further, until again it is at a local minimum. This is the centre point.

This strategy is unrealistic, because continuously lowering the volume is time-consuming, and also transceivers typically have only 5-10 volume settings. We improve upon the traditional strategy using volume settings. Refer to Figure 6.

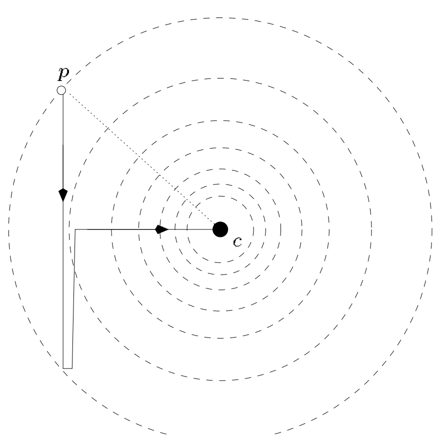

Figure 6: Traditional strategy with multiple volumes.

1. Leave a marker at point p, and find a direction from p that leads inside circle C_1.
2. Start moving along the chord, but periodically lower the volume and mark the location so that the signal can just be heard. If your volume is at the lowest level you are in the innermost circle and stop.
3. Move until the signal fades. Mark this point.
4. Move to the midpoint of the last two markers.
5. Turn $90°$ to the direction of the centre.
6. Repeat steps 3-5.

The efficiency of this strategy depends greatly on the number of circles and how much the radii decrease. The distance decreases as the number of circles increase. Assuming the outermost circle has a radius of 30m and the innermost has a radius of one meter, one can show that the competitive ratio is no worse than 3.16 with five volume settings. Doubling the number of settings to ten does not help much, as the ratio decreases to 2.84.

This strategy has two major disadvantages. First, it requires many markers, proportional to the number of volume settings. Second, the searcher may miss the innermost circle, causing the searcher to start another phase of searching from the boundary of the same circle.

4 A robust strategy

If we allow neither volume nor markers to be used, then we will not be able to locate the centre-point, because no information can be stored, and so we will never know anything more than whether we are inside circle C.

However, even without using markers, which means in turn that we cannot determine midpoints of segments, we can locate the innermost circle, provided that the radii between consecutive circles decreases by less than $1/\sqrt{2}$. If this condition holds, then by traversing two consecutive chords at right angles, we must intersect a smaller circle. The precise strategy is as follows:

1. Move along the first chord as before, but lower the volume whenever possible.
2. When the signal fades, turn $90°$ and move along the next chord. If the signal fades right away, turn $180°$ and move in the opposite direction.
3. Repeat the above steps until you have reached the lowest volume setting.

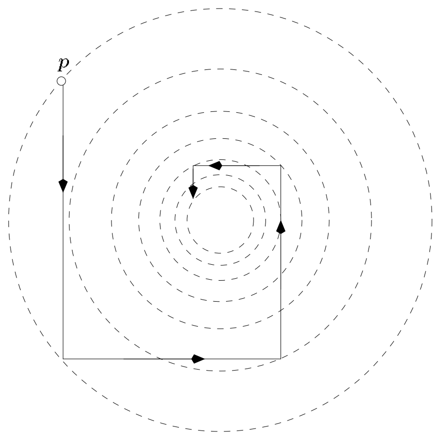

Figure 7: Doing right turns finds the smallest circle.

Analyzing the competitive ratio of this strategy is not straightforward, since the length of the path depends on where we intersect the next smaller circle. If the radii of consecutive circles decrease by $1/\sqrt{2}$, the competitive ratio is at most 5.535. Not having to leave markers may be advantageous, and finding midpoints seems more difficult than making right turns, which means that in practice this strategy might be very effective.

5 Open Problems

Two main problems remain. First, we assumed that induction lines from the transceivers are circles. In reality, they resemble ellipses, and have "dead" areas where no signal is received even though the origin is close. How can this be handled? Secondly, experiments in real-life ski settings should be conducted to compare the effectiveness and ease of learning of the right-angle strategy (or any of our suggested improvements) as compared to the traditional strategy. This certainly must be done before anyone attempts this method in a life or death situation.

There is another strategy by searchers, using more modern transceivers which give some directional information. By testing every few meters, the searcher can follow an induction line which curves towards the sending transceiver. These transceivers even can estimate the distance to the sender [5]. It would be interesting to develop theoretically better search strategies for this scenario as well.

Another question is whether the right angle strategy can be used in the roundness classification problem of [2]. One might be able to show that in their theorem 2, only 5 probes are needed to determine an approximate centre of an almost round object instead of 6. Three probes forming a right triangle can replace four probes described in their technique, which like the traditional skiing strategy, forms two chords which cross at right angles.

6 Acknowledgments

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References

[1] David George, "The Survival Curve", *French off-piste skiing website* (http://www.PisteHors.com), 2002.

[2] Kurt Mehlhorn, Thomas C. Shermer, and Chee K. Yap, "A complete roundness classification scheme,*13th ACM Symposium on Computational Geometry*, 1997, 129-138.

[3] Ronald I. Perla and M. Martinelli Jr. *Avalanche Handbook.* U.S. Department of Agriculture, Forest Service, July 1976.

[4] Michael G. Smith. "Avalanche Transceiver Recovery Skills". *GearWorld* (http://www.gearworld.com), 2002.

[5] http://www.bcaccess.com/ftrax.html, 2002.