Lower Bounds for Streets and Generalized Streets*

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ABSTRACT

We present lower bounds for on-line searching problems in two special classes of simple polygons called streets and generalized streets. In streets we asssume that the location of the target is known to the robot in advance and prove a lower bound of $\sqrt{2}$ on the competitive ratio of any deterministic search strategy which can be shown to be tight.

For generalized streets we show that if the location of the target is not known, then there is a class of orthogonal generalized streets for which the competitive ratio of any search strategy is at least $\sqrt{82} \sim 9.06$ in the L_2-metric—again matching the competitive ratio of the best known algorithm. We also show that if the location of the target is known, then the competitive ratio for searching in generalized streets in the L_1-metric is at least 9 which also can be shown to be tight.

The former result is based on a lower bound on the average competitive ratio of searching on the real line if an upper bound of D to the target is given. We show that in this case the avervage competitive ratio is at least $9 - O(1/\log D)$.

1. Introduction

The problem of a robot searching for a target in an unknown environment has recently received a considerable amount of attention [3,4,5,6,11,12,15,17,18]. In this setting it is assumed that the robot is equipped with an on-board vision system that allows it to see its local environment. However, the robot does not have access to a complete map of its surroundings.

Since the robot has to make decisions about the search based only on the part of its environment that it has seen before, the search of the robot can be viewed as an *on-line* problem. One way to judge the performance of an on-line search strategy is to compare the distance traveled by the robot to the length of the shortest path from s to t. In other words, the robot's path is compared with that of an adversary who knows the complete environment; this approach to analysing on-line algorithms was introduced by Sleator and Tarjan [20]. The ratio of the distance traveled by the robot to the optimal distance from s to t is called the *competitive ratio* of the search strategy.

Since, in general, the ratio between the distance a robot traverses and the length of a shortest path can be forced to be $\Omega(n)$ if the obstacles in the scene have a total of n edges, efforts have focussed on restricted classes of environments that allow more efficient search strategies.

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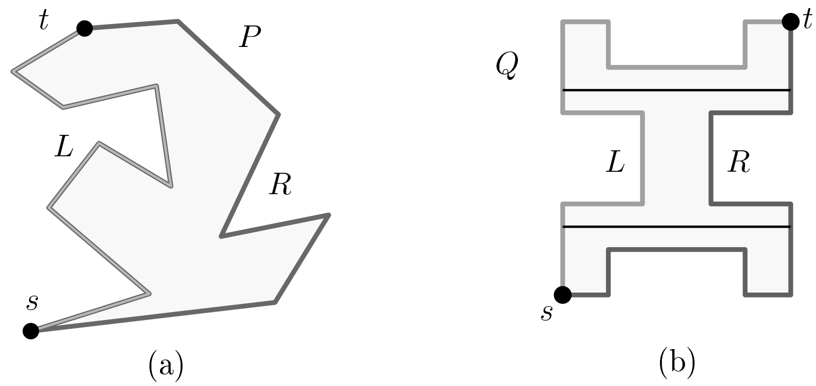

Figure 1: (a) The triple (P, s, t) is a street since the two polygonal chains L and R from s to t are weakly visible from each other. (b) The triple (Q, s, t) is a $\mathcal{G}$-street (but not a street) since every point of Q can be seen by the two horizontal chords that connect L and R.

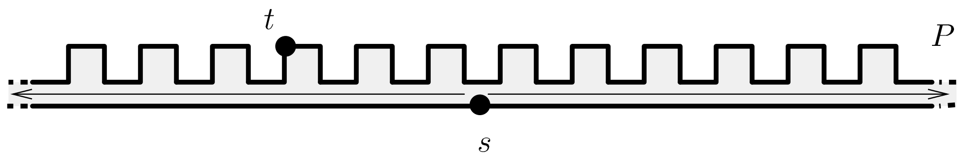

Figure 2: A lower bound for $\mathcal{G}$-streets.

Klein was the first to consider a class of polygons called *streets* [11]. A *street* is a triple (P, s, t) where P is a simple polygon and s and t are two points on the boundary of P such that the two polygonal chains L and R from s to t are mutally weakly visible, that is, each point on L sees a point on R and vice versa (see Figure 1a). It can be easily shown that $\sqrt{2}$ is a lower bound for the competitive ratio of searching in an orthogonal street. Surprisingly, this competitive ratio can be achieved even for arbitrarily oriented streets [9,19]. In this paper, we show that $\sqrt{2}$ remains a lower bound for searching in (orthogonal) streets even if the location of the target is known in advance. However, streets are often too restrictive a class of polygons in order to model real environments.

In the search for larger classes of polygons that admit search strategies with a constant competitive ratio Datta and Icking propose a class of polygons they call *generalized streets* or *$\mathcal{G}$-streets* [6]. Similar to a street, a $\mathcal{G}$-street is also a triple (P, s, t) but now each point on L or R has to be seen from a *horizontal chord* that connects the two polygonal chains L and R (see Figure 1b). It can be shown that the class of $\mathcal{G}$-streets properly contains the class of streets. Datta and Icking present an algorithm to search in orthogonal $\mathcal{G}$-streets that achieves a competitive ratio of 9 in the L_1-metric and of $\sqrt{82}$ (~ 9.06) in the L_2-metric. In fact, 9 is also a lower bound for searching in orthogonal $\mathcal{G}$-streets. To see this consider the polygon P in Figure 2. If the target t is placed in any of the spikes, then (P, s, t) can be easily shown to be a $\mathcal{G}$-street. Obviously, a strategy to search in P can be used to search for a target on the real line (without visibility information). Since 9 is a lower bound for searching on the line [1,7,14], we obtain the same lower bound for searching in $\mathcal{G}$-streets.

López-Ortiz and Schuierer present an algorithm to search in arbitrarily oriented $\mathcal{G}$-streets albeit with a much higher competitive ratio [16].

In this paper we present a family of orthogonal $\mathcal{G}$-streets in which every search strategy needs a competitive ratio of at least $\sqrt{82}$ if the path length is measured in the L_2-metric, thus, proving that the strategy of Datta and Icking is optimal even in the L_2-metric.

Finally, we consider the problem of searching in an orthogonal $\mathcal{G}$-street if the location of the target is known in advance. We again show that this knowledge does not provide an advantage to the robot and show a lower bound of 9 in the L_1-metric.[1]

In order to be able to prove the lower bound of 9 for known destination search in $\mathcal{G}$-streets we use a reduction to a variant of searching for a target on the real line. In this new variant we consider the average competitive ratio C^{av} of a strategy where the average is taken over the competitive ratio C^L if the target is found to the left of the starting position and the competitive ratio C^R if the target is found to the right of the starting position, that is, $C^{av} = (C^L + C^R)/2$; moreover, we assume that we are given an upper bound D on the maximal distance of the target to the starting position. We show that $C^{av} \geq 9 - O(1/\log D)$. This new bound may be of use in other lower bound proofs of similar nature.

The paper is organized as follows. In Section 2 we show a lower bound of $\sqrt{2}$ for searching in streets given the location of the target in advance. In Section 3 we present lower bounds for orthogonal $\mathcal{G}$-streets with known and unknown location of the target. In Section 4 we prove a lower bound for biased search strategies on the real line. Finally, in Section 5 we summarize our results.

2. A Lower Bound for Searching in Streets

In this section we present a lower bound for searching in streets. If we assume that the position of the target is not known to the robot in the beginning, then the polygon shown in Figure 3a[2] provides a lower bound of $\sqrt{2}$. If the robot goes to one side, before it can see into both ears of the polygon at the top of the rectangle, then the target is placed in the ear on the other side. Hence, going straight up in the middle until both ears are visible is the best on-line strategy. If we choose the width to be 2 and the height to be 1, then the competitive ratio is $\sqrt{2}$.

The question is if there is a strategy with a better competitive ratio if the location of the target is known. Clearly, the polygon shown in Figure 3a now no longer provides a lower bound since the robot knows the position of the target and can move directly to it; however, by connecting a number of these polygons, it is still possible to show a lower bound of $\sqrt{2}$ on the competitive ratio of any strategy to search in streets even if the position of the target is known in advance.

More precisely, we construct a family $\mathcal{F}_n$ of orthogonal streets such that, for all $n \geq 0$ and for all on-line strategies S, there is a street P_S in $\mathcal{F}_n$ such that the competitive ratio of S in P_S is $\sqrt{2} - O(1/\sqrt{n})$. Note that the restriction to orthogonal streets only strengthens our lower bound.

[1]Preliminary versions of the lower bounds for streets and $\mathcal{G}$-streets have appeared in [15] and [16].
[2]For now, the paths drawn in the figure should be disregarded.

We call the polygon of Figure 3a an *eared-rectangle*. Eared-rectangles can be connected to create larger polygons. This is shown in Figure 3b. In the construction

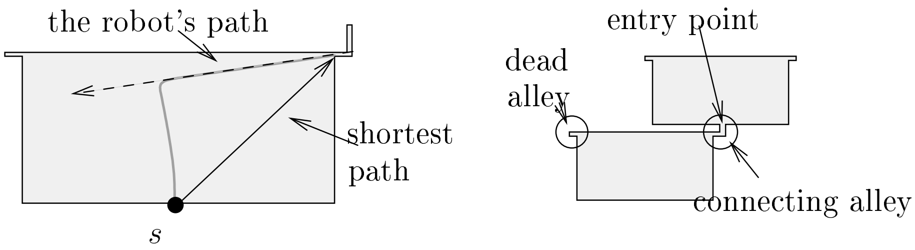

Figure 3: (a) An eared-rectangle. (b) Connecting two eared-rectangles

of Figure 3 each eared-rectangle has a *connecting alley* and a *dead-alley*. The *entry point* is the point where the robot enters an eared-rectangle which is located in the middle of its bottom edge.

If the robot is located inside an eared-rectangle and wants to decide which of the two alleys is dead and which is connecting, then it has to move up to a point in the eared-rectangle from which the top edge of one of the alleys is completely visible. By making the alleys very narrow, we can force the robot to move arbitrarily close to the horizontal line that connects the alleys before it can decide which alley is dead and which is connecting.

Assume we are given a strategy S to search in an orthogonal street with known destination. In the beginning the robot is located in an eared-rectangle of width 2 units and height 1 unit. The target is located directly above the starting point s at a distance of n units.

We present the strategy of an *adversary* to S that constructs a polygon consisting of at most $n^2/2$ connected eared-rectangles in which the path traversed by the robot using S is at least $\sqrt{2} - O(1/\sqrt{n})$ times longer than the shortest path from s to t.

The adversary's strategy is as follows. If the robot moves into the left half of the eared-rectangle in order to find out which alley is connecting, then the adversary opens the right alley and connects a new eared-rectangle to it and vice versa. If the robot travels in the middle of the eared-rectangle, then the adversary opens an arbitrary alley. In this way the length of the path generated by S in one eared-rectangle has a length of at least $2 - \varepsilon$ where ε depends on the width of the alleys whereas the shortest path has a length of $\sqrt{2}$ (see Figure 3a).

The adversary puts one eared-rectangle on top of the other until n rectangles have been placed and the top edge of the current eared-rectangle has the same height as t. In this case the next eared-rectangle is rotated by $90°$ and placed on the side of the current eared-rectangle that is closer to t. We denote the entry point of this rotated eared-rectangle by s_2 (see Figure 4a).

First of all we note that at s_2 the situation is exactly analogous to the situation at s just rotated by $90°$. This is due to the fact that the shaded region in Figure 4b does not contain any eared-rectangles and the target t is again on an axis-parallel line through s_2. Hence, the adversary can apply the same strategy recursively now starting at s_2. Since the distance of s_2 to t is at least one less than the distance of

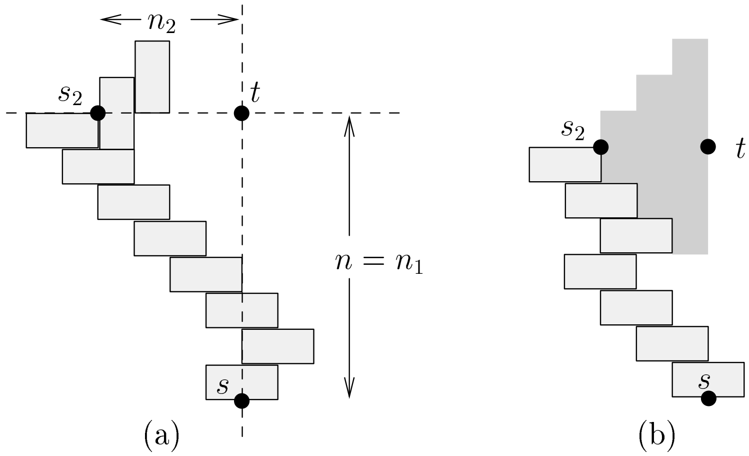

Figure 4: (a) Constructing a new polygon with eared-rectangles. (b) The situation at s_2 is analogous to the situation at s.

s to t, this construction ends after at most n iterations. Clearly, if P is the polygon so constucted, then (P, s, t) is a street. Let k be the actual number of iterations needed. We denote the starting point of the ith iteration s_i, for $1 \leq i \leq k$, where $s_1 = s$.

We now analyse the distance traveled by the robot. As we observed above, the length of the path generated by S in one eared-rectangle is at least $2 - \varepsilon$ units whereas the length of the shortest path is $\sqrt{2}$ units. This is true for all eared-rectangles except for the last eared-rectangle of an iteration whose top edge has the same height as t. In this case the action of the adversary does not depend on S, but the adversary always rotates the new eared-rectangle and opens the alley that is closer to t. We assume that S is given this knowledge in advance and, hence, S is able to choose the shortest path in the last eared-rectangle of an iteration. Note that if the distance of s_i to t is n_i, then the adversary places n_i eared-rectangles on top of each other until the horizontal or vertical line through t is reached. Hence, the distance traveled by the robot in the ith iteration is $(n_i - 1)(2 - \varepsilon) + \sqrt{2}$ whereas the length of the shortest path is $n_i\sqrt{2}$. The competitive ratio of S is now at least

$$
\begin{aligned}
\frac{\sum_{i=1}^{k} \left((n_i - 1)(2 - \varepsilon) + \sqrt{2}\right)}{\sqrt{2}\sum_{i=1}^{k} n_i} &= \left(\sqrt{2} - \frac{\varepsilon}{\sqrt{2}}\right) - \frac{(2 - \sqrt{2} - \varepsilon)k}{\sqrt{2}\sum_{i=1}^{k} n_i} \\
&\geq \left(\sqrt{2} - \frac{\varepsilon}{\sqrt{2}}\right) - \frac{k}{\sum_{i=1}^{k} n_i} \qquad (1)
\end{aligned}
$$

with $1 \leq n_k < n_{k-1} < \cdots < n_2 < n_1 = n$. The Strategy S can choose the numbers k and n_i, for $1 \leq i \leq k$, in order to minimize Expression 1. It is minimized if $\sum_{i=1}^{k} n_i$ is as small as possible, that is, if $n_k = 1$, $n_{k-1} = 2$, and so on until $n_2 = k - 1$ and $n_1 = n$. Therefore, Expression 1 is bounded by

$$
\left(\sqrt{2} - \frac{\varepsilon}{\sqrt{2}}\right) - \frac{k}{\sum_{i=1}^{k-1} i + n} = \left(\sqrt{2} - \frac{\varepsilon}{\sqrt{2}}\right) - \frac{k}{(k-1)k/2 + n}.
$$

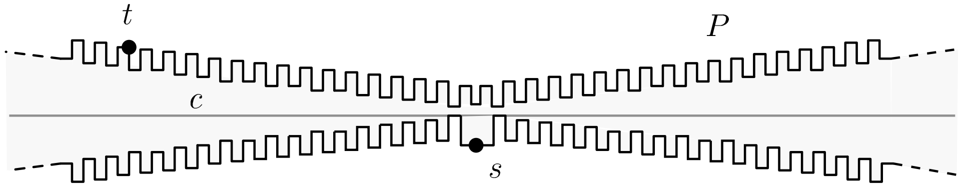

Figure 5: A $\mathcal{G}$-street which forces a competitive ratio of $\sqrt{82}$.

This is minimized for $k = \sqrt{2n}$, and the competitive ratio of S is at least

$$\sqrt{2} - \frac{\varepsilon}{\sqrt{2}} - \frac{1}{\sqrt{2n} - 1/2} \leq \sqrt{2} - \frac{\varepsilon}{\sqrt{2}} - \frac{1}{\sqrt{n}},$$

for $n \geq 2$. By choosing the alleys so small that $\varepsilon = 1/\sqrt{n}$, the claim follows. Since n can be arbitrarily large, we have shown the following result.
Theorem 1 *If S is a deterministic strategy to search in streets with known location of the target, then the competitive ratio of S is at least $\sqrt{2}$.*

3. Lower Bounds in $\mathcal{G}$-streets

In this section we prove lower bounds for two types of search problems in $\mathcal{G}$-streets. First we show that the competitive ratio of searching in a rectilinear $\mathcal{G}$-street is at least $\sqrt{82} \sim 9.06$ if the length of a path is measured in the L_2-metric. Secondly, we show that 9 remains a lower bound to search in rectilinear $\mathcal{G}$-streets if the length of a path is measured in the L_1-metric even if the coordinates of the target are known in advance.

3.1. A Lower Bound for the L_2-Distance

Consider the $\mathcal{G}$-street P in Figure 5. The target t can be hidden in any of the teeth of P and P still is a $\mathcal{G}$-street. In order to decide whether the target t is contained in a tooth T, the robot must intersect the vertical line through the rightmost point of T if T is to the left of s and the vertical line through the leftmost point of T if T is to the right of s. If P contains n teeth, then the robot can be forced to travel at least $9 - O(1/\log^2 n)$ times the horizontal distance of s to the tooth that contains t [1,14]. It does not pay for the robot to leave the chord c; since if the robot is located above c when it detects the target, then an adversary places t in a tooth below c and vice versa. If p is the point on c at which the robot sees t, then the robot travels a distance of $9d(s,p) + d(p,t)$ while the L_2-shortest path has length $\sqrt{d(s,p)^2 + d(p,t)^2}$. By choosing $d(p,t) = 1/9d(s,p)$, i.e., by putting the teeth of P along lines with slopes $1/9$ and $-1/9$, respectively, we obtain a competitive ratio of $\sqrt{82}$ for $n \to \infty$ as claimed. We have shown the following theorem.
Theorem 2 *The competitive ratio of any search strategy to search in orthgonal streets is at least $\sqrt{82} \sim 9.06$ in the L_2-metric.*

3.2. A Lower Bound for Searching for a Target of Known Location

Now consider the situation in which the robot searches for a target of known

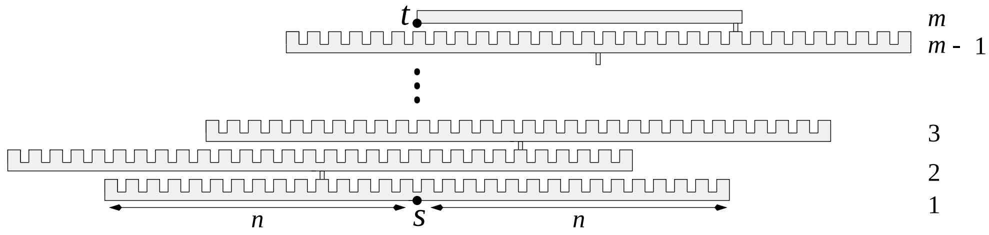

Figure 6: A Lego-stack polygon.

location on a $\mathcal{G}$-street. In this case the polygon of Figure 5 no longer provides a lower bound of 9. Instead, similar to the lower bound for searching in streets we again construct a family $\mathcal{F}_n$ of polygons such that for each deterministic search strategy S there is a polygon P_S in $\mathcal{F}_n$ for which a robot using S traverses at distance of at least $9 - O(1/\log n)$ times the length of a shortest path from s to t.

Theorem 3 *Searching for a target of known location in a rectilinear $\mathcal{G}$-street is at least 9-competitive.*

In the following we prove Theorem 3. Let the origin be the initial position s of the robot and $(0, 1)$ the position of the target t. Each polygon in the family $\mathcal{F}_n$ of Lego-stack polygons is made of $m = n^3$ connected rake polygons. A connection point joins a tooth from the bottom rake to the middle of the top rake (see Figure 6).

Rakes are numbered in the order of occurrence on the robot's path from s to t. Each rake has height $1/m$; it is symmetrically centered above its entrance point and has length $2n$ (except for the last one which contains no teeth and is just wide enough to contain t and its entrance point). At the entrance point the robot sees only into one tooth of the rake. It searches for the opening to the next rake by alternatingly exploring the parts of the rake to the left and right of the entrance point in increasing step lengths. We define C_k^L to be the competitive ratio of the search strategy of the robot if the target is found in the kth exploration of the left side. C_k^R is defined analogously for the right side.

The problem in the construction of the Lego-stack polygon is that since the robot knows the location of the target it can bias the search towards to target. However, we can make use of the following theorem which ensures that the more the robot biases the search towards the target, the higher is the penalty if the adversary places the connecting tooth on the side opposite to the target. In the following let $C_n = 9 - 72/(\log n - 8)$.

Theorem 4 *If X is a strategy to search for a target in a rake of length $2n$ having a total of $2l$ steps, then*

$$\max_{1 \le k \le l} (C_k^L + C_k^R)/2 \ge C_n.$$

This theorem is proven in Section 4.

3.2.1. *

Adversary's Strategy

We use an adversary to construct the polygon P_S on-line depending on the robot's moves. Let x_i be the x-coordinate of the entrance point to the ith rake and

d_i the horizontal distance from the entrance point to the exit (connecting) tooth, that is, $d_i = |x_i - x_{i+1}|$.

The adversary keeps track of x_i, the competitive ratios C^L and C^R, and a variable D_i. The competitive ratio C^L is defined as the ratio of the distance that the robot has traversed *in the current rake* over the distance to the closest unexplored tooth in the left part. C^R is defined analogously for the right side. D_i is defined as the length of a shortest path from s to the entrance point of rake i minus $|x_i|$.

The adversary's strategy to place rake $i+1$ works as follows. We assume that $x_i \geq 0$. The case $x_i < 0$ is completely analogous. The adversary first checks if there is a step in the exploration of rake i such that C^L is not too small, that is, such that $C^L \geq 2C_n - c_{k_i+1}$ where $k_i = \lfloor |x_i|/n \rfloor$ and c_k is defined by

$$c_k = \begin{cases} C_n, & \text{for } k \leq 0 \\ C_n + (k-1)/n, & \text{for } k > 0. \end{cases}$$

If this is the case, then adversary opens the tooth with competitive ratio C^L and sets $x_{i+1} = x_i - d_i$. The length of the shortest path from s to the entrance point of rake $i+1$ increases by d_i, that is, by the definition of D_i its length is $x_i + D_i + d_i$ and $D_{i+1} = x_i + D_i + d_i - |x_{i+1}|$. If we distinguish the cases $x_i \geq d_i$ and $x_i < d_i$, then it is easy to see that the equation for D_{i+1} simplifies to $D_{i+1} = D_i + 2\min\{d_i, x_i\}$.

If the competitive ratio C^L is smaller than $2C_n - c_{k_i+1}$ for all steps of the exploration of rake i, then the adversary opens a tooth to the right of the entrance point with competitive ratio $C^R \geq 2C_n - C^L$ which is always possible by Theorem 4. We obtain $x_{i+1} = x_i + d_i$ and $D_{i+1} = D_i$. In a more algorithmic notation the adversary's strategy can be described as follows.

Adversary's Strategy

$D_1 \leftarrow 0$; $x_1 \leftarrow 0$
for $i \leftarrow 1$ **to** $m-1$ **do**
  $k_i \leftarrow \lfloor |x_i|/n \rfloor$
  if $x_i \geq 0$ **then**
    /* The robot is to the right of s */
    if there is a step such that $C^L \geq 2C_n - c_{k_i+1}$
      then the adversary opens the tooth to the left of the entrance point with competitive ratio C^L
        $x_{i+1} \leftarrow x_i - d_i$
        $D_{i+1} \leftarrow D_i + 2\min\{d_i, x_i\}$
      else the adversary opens a tooth to the right of the entrance point with $(C^L + C^R)/2 \geq C_n$
        $x_{i+1} \leftarrow x_i + d_i$
        $D_{i+1} \leftarrow D_i$
  if $x_i < 0$ **then** ...
    /* this case is completely analogous to the case $x_i \geq 0$ with left and right (and plus and minus for x_i) exchanged */
end for

Since the competitive ratios C^L and C^R are non-negative, a tooth to the left of the entrance point is opened if it is ever the case that $2C_n - c_{k_i+1} \leq 0$, which implies that when a tooth to the left of the entrance point is opened we have $\lfloor x_i/n \rfloor \leq nC_n$ and if a tooth to the right of the entrance point is opened if $\lfloor -x_i/n \rfloor \leq nC_n$. This implies the following observation.

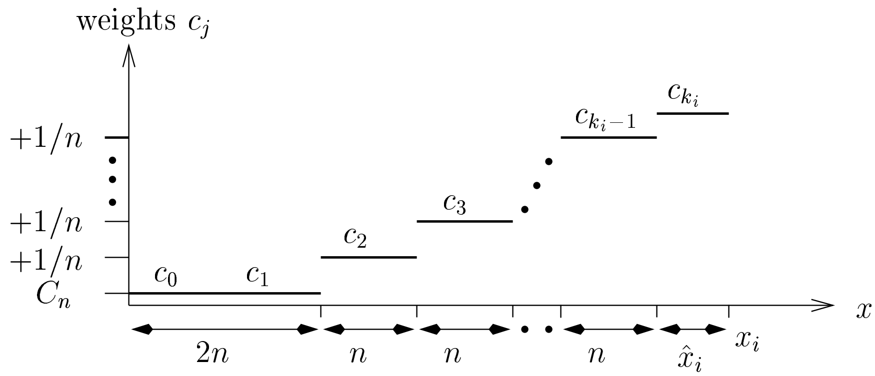

Figure 7: Each block of length n that fits into $|x_i|$ is weighted by the factors c_j.

Observation 1

$$|x_i| \leq (nC_n + 1)n \leq 10\, n^2.$$

In order to compute the distance traversed by the robot we show that the following invariant is maintained during the construction of P_S.

Invariant 1 *When entering rake i the robot has traversed a distance of at least*

$$L_i = \left(\sum_{j=0}^{k_i - 1} c_j \right) n + c_{k_i} \hat{x}_i + \left(C_n - \frac{1}{n} \right) D_i,$$

where $\hat{x}_i = |x_i| - k_i n$.

Invariant 1 states that each block of length n that fits into $|x_i|$ is weighted by the factors c_j which increase by $1/n$ (except for c_1 which equals c_0). Hence, the larger $|x_i|$ the larger is the detour of the robot. This is illustrated in Figure 7.

Proof. W □

e show that the invariant holds after each construction step. The invariant obviously holds for $i = 1$. Now assume it holds up to (and including) iteration $i \geq 1$. For simplicity we assume that $x_i \geq 0$ in the following. The case $x_i < 0$ is completely analogous.

First consider Steps 6–8. By the invariant the total distance traveled by the robot is at least $L_i + C^L d_i$. First we assume that $x_i \geq d_i$, that is, $D_{i+1} = D_i + 2d_i$. Since $C^L \geq 2C_n - c_{k_i+1}$, the robot has traversed a distance of at least

$$L_i + (2C_n - c_{k_i+1})d_i = \left(\sum_{j=0}^{k_i - 1} c_j \right) n + c_{k_i} \hat{x}_i - \left(c_{k_i+1} - \frac{2}{n} \right) d_i + \left(C_n - \frac{1}{n} \right) \underbrace{(D_i + 2d_i)}_{= D_{i+1}}$$

Note that $c_{k_i} \geq c_{k_i - 1} \geq c_{k_i+1} - 2/n$. If $d_i \leq \hat{x}_i$, then $\hat{x}_{i+1} = \hat{x}_i - d_i$, $k_{i+1} = k_i$, and

$$c_{k_i} \hat{x}_i - \left(c_{k_i+1} - \frac{2}{n} \right) d_i \geq c_{k_i} \hat{x}_{i+1} = c_{k_{i+1}} \hat{x}_{i+1}.$$

If $d_i > \hat{x}_i$, then $\hat{x}_{i+1} = \hat{x}_i + n - d_i$, $k_{i+1} = k_i - 1$, and

$$c_{k_i-1}n + c_{k_i}\hat{x}_i - \left(c_{k_i+1} - \frac{2}{n}\right) d_i \geq c_{k_i-1}\hat{x}_{i+1} = c_{k_{i+1}}\hat{x}_{i+1}.$$

Hence,

$$L_i + (2C_n - c_{k_i+1})d_i \quad \geq \quad \left(\sum_{j=0}^{k_{i+1}-1} c_j\right) n + c_{k_{i+1}}\hat{x}_{i+1} + \left(C_n - \frac{1}{n}\right) D_{i+1}.$$

Now assume that $x_i < d_i$. In this case $k_i = 0$ and $\hat{x}_i = x_i$. Since $c_0 = c_1 = C_n$, the total distance traversed by the robot is at least

$$\begin{aligned} L_i + C^L d_i \quad &\geq \quad \underbrace{c_0}_{=C_n} x_i + \left(C_n - \frac{1}{n}\right) D_i + \underbrace{(2C_n - c_1)}_{=C_n} d_i \\ &\geq \quad C_n(d_i - x_i) + \left(C_n - \frac{1}{n}\right)(D_i + 2x_i) \quad = \quad c_0|x_{i+1}| + \left(C_n - \frac{1}{n}\right) D_{i+1}. \end{aligned}$$

Now consider Steps 9–11. As we observed above, Theorem 4 implies that there is a step with $(C^L + C^R)/2 \geq C_n$. Since for all steps $C^L < 2C_n - c_{k_i+1}$, we obtain

$$C^R \geq 2C_n - C^L > c_{k_i+1} \geq c_{k_{i+1}}.$$

Hence, the distance traveled by the robot in iteration i is $C^R d_i \geq c_{k_{i+1}} d_i$ and since $x_{i+1} = x_i + d_i$, Invariant 1 is clearly maintained.

3.2.2. *

The Competitive Ratio

We now analyse the competitive ratio of a strategy for which a Lego-stack polygon is constructed in the above manner. In the mth rake the target has distance $|x_m|$ from the entrance point. Hence, the total distance traveled by the robot is at least $\left(\sum_{j=0}^{k_m-1} c_j\right) n + c_{k_m}\hat{x}_m + \left(C_n - \frac{1}{n}\right) D_m + |x_m|$ and the competitive ratio C of the strategy is bounded from below by

$$\frac{\left(\sum_{j=0}^{k_m-1} c_j\right) n + c_{k_m}\hat{x}_m + \left(C_n - \frac{1}{n}\right) D_m + |x_m|}{D_m + 2|x_m|}.$$

We observe that $D_m \geq m - 2|x_m|$ since the robot moves at least one step in each iteration and, thus, the length of a shortest path from s to t is $D_m + 2|x_m| \geq m$. Hence, by using the taylor series expansion for $1/(1+x)$, Observation 1 and the fact that $m = n^3$ we obtain

$$\begin{aligned} C \quad &\geq \quad \frac{\left(C_n - \frac{1}{n}\right) D_m}{D_m + 2|x_m|} \geq \left(C_n - \frac{1}{n}\right) \frac{1}{1 + \frac{2|x_m|}{D_m}} \geq \left(C_n - \frac{1}{n}\right)\left(1 - \frac{2|x_m|}{D_m}\right) \\ &\geq \quad \left(C_n - \frac{1}{n}\right)\left(1 - \frac{20n^2}{n^3 - 20n^2}\right) = C_n - O\left(\frac{1}{n}\right) = 9 - O\left(\frac{1}{\log n}\right), \end{aligned}$$

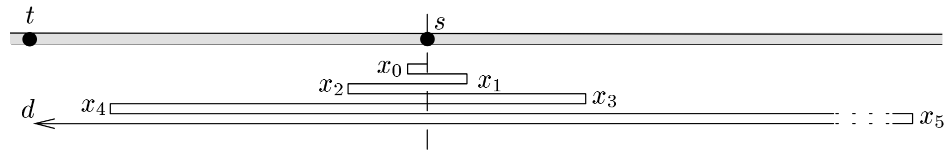

Figure 8: Searching on the real line.

for $n > 20$. This concludes the proof.

4. Searching on the Real Line

In this section we prove Theorem 4. In fact, we consider the more general setting of searching on the real line. Here, we assume that the robot is located at the origin s and the target t is located elsewhere on the line. The robot can only detect t if it stands on top of it. We assume that we are given a lower bound of 1 distance unit on the minimal and an upper bound of D distance units on the maximal distance to the target.

A strategy to search for t works as follows. The robot starts at s and travels to one side, say to the left. At some point, say at a distance of x_0 to s, it decides that it has traveled far enough to the left and returns to s. It travels a distance of x_1 to the right and returns to s again to explore the left side again and so on. For illustration see Figure **??**. Obviously, the values x_i which denote the distance that the robot travels to the left or to the right of s—depending on whether i is even or odd—suffice to characterize a search strategy completely. Note that since an upper bound of D on the distance to t is known, the strategy consists of a finite number of steps which we assume to be $(x_0, \ldots, x_{2n+1})$.[3]

4.0.3. *

The Competitive Ratio

Assume that the target is discovered in Step $k+2$, say to the left of the origin. The ray to the left of the origin was visited the last time before Step $k+2$ in Step k. Hence, the distance d of the target is greater than x_k. The distance traveled by the robot to discover t is $d + 2\sum_{i=0}^{k+1} x_i$. The *competitive ratio of Step k* is given by

$$\sup_{d > x_k} \frac{d + 2\sum_{i=0}^{k+1} x_i}{d} = 1 + 2\frac{\sum_{i=0}^{k+1} x_i}{x_k}$$

since d can be placed arbitrarily close to x_k by an adversary. The competitive ratio C of the strategy is now given as the maximum of the competitive ratios of the Steps k with $0 \le k \le 2n$, that is, $C = \max_{0 \le k \le 2n} 1 + 2(\sum_{i=0}^{k+1} x_i)/x_k$. It can be shown that, if $D = \infty$, then the competitive ratio of the strategy is minimized if $x_i = 2^i$ [1,2,7]. This results in a competitive ratio of 9.

[3] Note that the number $2n+2$ of steps of a strategy is not related to the width $2n$ of a rake which is a distance measurement and corresponds to D. The reason why we choose the number of steps to be $2n+2$ instead of n is due to the fact that we will work with sequences that have about half as many elements as X.

4.0.4. *

Biased Strategies

We say a search strategy is *biased* if one side is favored over the other. For instance, the robot may explore the ray to the left much farther than the ray to the right. Of course, the overall competitive ratio of such a strategy is at least 9 as mentioned above. However, suppose that the competitive ratios for the left and right sides are considered separately. We define the *left competitive ratio L_k of Step $2k$* to be the competitive ratio if the target is placed on the left side and is found in Step $2k+2$; analogously, we define the *right competitive ratio R_k of Step $2k+1$*; that is,

$$L_k = 1 + 2\frac{\sum_{i=0}^{2k+1} x_i}{x_{2k}} \qquad \text{and } R_k = 1 + 2\frac{\sum_{i=0}^{2k+2} x_i}{x_{2k+1}}.$$

Let $C^{av} = \max_{0\le k\le n-1}(L_k + R_k)/2$ which we call the *average competitive ratio* of X. Within this framework Theorem 4 can now be stated as follows.

Theorem 4′ *If X is a strategy to search for a target on the real line whose distance is at most D to s and C_D^{av} is its average competitive ratio, then $C_D^{av} \ge 9 - 72/(\log D - 8)$.*

We use the following approach to prove a lower bound on C_D^{av}. Assume that strategy X consists of $2n+2$ steps and the average competitive ratio of X is less than 9 (otherwise the theorem is trivially true). Let $\widehat{C}_{2n+2}$ be twice the minimal average competitive ratio that a positive sequence consisting of $2n+2$ elements can achieve. In particular, $C_D^{av} \ge \widehat{C}_{2n+2}/2$. We compute a lower bound C_n^* on $\widehat{C}_{2n+2}$. Then, we show that $C_n^* \ge 18 - 36/n$ which implies that $C_D^{av} \ge 9 - 18/n$.

Finally, we observe that since

$$8 \ge \frac{L_k + R_k}{2} - 1 = \frac{\sum_{i=0}^{2k+1} x_i}{x_{2k}} + \frac{\sum_{i=0}^{2k+2} x_i}{x_{2k+1}} \ge \frac{x_{2k+1}}{x_{2k}} + \frac{x_{2k+2}}{x_{2k+1}},$$

for $0 \le k \le n-1$, the ratio x_{2k+2}/x_{2k} is bounded by 16 and, therefore, $D \le x_{2n} \le 16^{n+1}x_0$. Hence, $\log D \le 4(n+1) + \log x_0 \le 4(n+1) + 4$ and $C_D^{av} \ge 9 - 72/(\log D - 8)$ which proves Theorem 4′.

Hence, in the remaining section it is our aim to show the following lemma.

Lemma 2 *There is a C_n^* with $C_n^* \le \widehat{C}_{2n+2}$ such that*

$$C_n^* \ge 18 - \frac{36}{n}.$$

Once we have shown Lemma 2, Theorem 4′ follows by our above considerations. Since the proof of Lemma 2 is somewhat involved we show the claim in several steps which are formulated as lemmas.

4.0.5. *

Computing a Lower Bound on $\widehat{C}_{2n+2}$

In the following let X be a sequence consisting of $2n+2$ elements $(x_0, \ldots, x_{2n+1})$. We define $l_k = x_{2k}$ and $r_k = x_{2k+1}$, for $0 \le k \le n$. Furthermore, let $\overline{L}_k = \sum_{i=0}^{k} l_i$ and $\overline{R}_k = \sum_{i=0}^{k} r_k$. $\overline{L}_k$ is half the distance traversed on the left side and R_k is half the distance traversed on the right side. As above let L_k be the left competitive of Step $2k$ and R_k the right competitive ratio of Step $2k+1$, that is,

$$L_k = 1 + 2\frac{\sum_{j=0}^{2k+1} x_j}{x_{2k}} = 1 + 2\frac{\overline{L}_k + \overline{R}_k}{l_k}$$

and

$$R_k = 1 + 2\frac{\sum_{j=0}^{2k+2} x_j}{x_{2k+1}} = 1 + 2\frac{\overline{L}_{k+1} + \overline{R}_k}{r_k}.$$

If there is a $0 \le k \le n-1$ such that $L_k + R_k \ge 18$, then we are done. Hence, we assume in the following that $L_k + R_k < 18$, for all $0 \le k \le n-1$.

Lemma 3 *For all positive sequences X of length $2n+2$ with $L_k + R_k < 18$, for $0 \le k \le n-1$, there is a sequence $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_{n-1})$ with $\Omega_k \in [1/4, 12]$, for $1 \le k \le n-1$ such that*

$$\max_{0 \le k \le n-1} L_k + R_k \ge \max_{1 \le k \le n-2} 6 + \Omega_k + \frac{4}{\Omega_{k+1}} + 2\sqrt{2(\Omega_k + 2)\left(1 + \frac{2}{\Omega_{k+1}}\right)}. \quad (2)$$

Proof. L □

et L_k and R_k be defined as above. For $1 \le k \le n-1$, we set

$$\Omega_k = 2\frac{\overline{L}_{k-1} + \overline{R}_{k-1}}{l_k}.$$

The sequence $(\Omega_1, \ldots, \Omega_{n-1})$ is called the *residue sequence* of X. L_k can now be expressed as follows

$$L_k = 1 + 2\frac{\overline{L}_k + \overline{R}_k}{l_k} = 3 + 2\frac{\overline{L}_{k-1} + \overline{R}_{k-1}}{l_k} + \frac{2r_k}{l_k} = 3 + \Omega_k + \frac{2r_k}{l_k}$$

which implies that

$$r_k = (L_k - 3 - \Omega_k)\frac{l_k}{2}. \quad (3)$$

Moreover,

$$L_k = 1 + 2\frac{\overline{L}_k + \overline{R}_k}{l_k} \quad \Rightarrow \quad 2(\overline{L}_k + \overline{R}_k) = (L_k - 1)l_k \quad (4)$$

and

$$\Omega_{k+1} = 2\frac{\overline{L}_k + \overline{R}_k}{l_{k+1}} \stackrel{(4)}{=} (L_k - 1)\frac{l_k}{l_{k+1}} \quad \text{or} \quad \frac{l_{k+1}}{l_k} = \frac{L_k - 1}{\Omega_{k+1}}. \quad (5)$$

We first show that $\Omega_k \in [1/4, 12]$, for all $1 \le k \le n-1$. $\Omega_k \le 12$ follows immediately from $L_k + R_k \le 18$, $\Omega_k + 3 \le L_k$, and $3 \le R_k$. We show that $\Omega_k \ge 1/4$ by contradiction. So assume that $\Omega_k < 1/4$. Then,

$$1/4 \ge \Omega_k = 2\frac{\overline{L}_{k-1} + \overline{R}_{k-1}}{l_k}$$

which implies that $l_k \geq 8(\overline{L}_{k-1} + \overline{R}_{k-1})$ and, therefore,

$$\begin{aligned} L_{k-1} + R_{k-1} \; > \; R_{k-1} &= 1 + 2\frac{\overline{L}_k + \overline{R}_{k-1}}{r_{k-1}} = 1 + 2\frac{\overline{L}_{k-1} + l_k + \overline{R}_{k-1}}{r_{k-1}} \\ &\geq 1 + 2\frac{\overline{L}_{k-1} + \overline{R}_{k-1} + 8(\overline{L}_{k-1} + \overline{R}_{k-1})}{r_{k-1}} \\ &= 19 + 18\frac{\overline{L}_{k-1} + \overline{R}_{k-2}}{r_{k-1}} \; > \; 18 \end{aligned}$$

in contradiction to the assumption that $L_{k-1} + R_{k-1} < 18$.

Next we show Equation 2. From Equations 3, 4, and 5 we obtain that, for $1 \leq k \leq n-2$,

$$\begin{aligned} L_k + R_k &= L_k + 1 + 2\frac{\overline{L}_{k+1} + \overline{R}_k}{r_k} \\ &= L_k + 1 + 2\frac{l_{k+1} + \overline{L}_k + \overline{R}_k}{r_k} \\ &\overset{(3,4)}{=} L_k + 1 + \frac{2l_{k+1} + (L_k - 1)l_k}{(L_k - 3 - \Omega_k)\frac{l_k}{2}} \\ &= L_k + 1 + 2\frac{L_k - 1}{L_k - 3 - \Omega_k} + \frac{4l_{k+1}}{(L_k - 3 - \Omega_k)l_k} \\ &\overset{(5)}{=} L_k + 1 + 2\frac{L_k - 1}{L_k - 3 - \Omega_k} + \frac{4(L_k - 1)}{(L_k - 3 - \Omega_k)\Omega_{k+1}} \\ &= L_k + 1 + 2(L_k - 1)\frac{1 + 2/\Omega_{k+1}}{L_k - 3 - \Omega_k}. \end{aligned}$$

Let

$$f(L_k, \Omega_k, \Omega_{k+1}) = L_k + 1 + 2(L_k - 1)\frac{1 + 2/\Omega_{k+1}}{L_k - 3 - \Omega_k}.$$

Taking the derivative of f w.r.t. L_k yields

$$\frac{\partial f}{\partial L_k}(L_k, \Omega_k, \Omega_{k+1}) = 1 + 2\frac{1 + 2/\Omega_{k+1}}{L_k - 3 - \Omega_k}\left(1 - \frac{L_k - 1}{L_k - 3 - \Omega_k}\right)$$

Hence, f has (at most) two extrema in L_k at

$$3 + \Omega_k \pm \frac{2\sqrt{4\Omega_{k+1}^2 + 2\Omega_{k+1}^2\Omega_k + 8\Omega_{k+1} + 4\Omega_{k+1}\Omega_k}}{2\Omega_{k+1}}.$$

Since $f \to -\infty$ as $L_k \to -\infty$, $f \to \infty$ as $L_k \to \infty$, and f has a polar point at $L_k = 3 + \Omega_k$, one extremum which is contained in $(-\infty, 3 + \Omega_k)$ is a local maximum of f, and the other extremum which is contained in $(3 + \Omega_k, \infty)$ is a local minimum. Since by Equation 3 $L_k > 3 + \Omega_k$, we only need to consider the minimum of f in $(3 + \Omega_k, \infty)$. The value of L_k at the minimum is

$$L_k(\Omega_k, \Omega_{k+1}) = 3 + \Omega_k + \frac{2\sqrt{4\Omega_{k+1}^2 + 2\Omega_{k+1}^2\Omega_k + 8\Omega_{k+1} + 4\Omega_{k+1}\Omega_k}}{2\Omega_{k+1}}.$$

In particular, $L_k + R_k \geq f(L_k(\Omega_k, \Omega_{k+1}), \Omega_k, \Omega_{k+1})$, for $1 \leq k \leq n-2$. If we set $g(\Omega_k, \Omega_{k+1}) = f(L_k(\Omega_k, \Omega_{k+1}), \Omega_k, \Omega_{k+1})$, that is,

$$g(\Omega_k, \Omega_{k+1}) = 6 + \Omega_k + \frac{4}{\Omega_{k+1}} + 2\sqrt{2(\Omega_k + 2)\left(1 + \frac{2}{\Omega_{k+1}}\right)},$$

then, $\max_{0 \leq k \leq n-1} L_k + R_k \geq \max_{1 \leq k \leq n-2} g(\Omega_k, \Omega_{k+1})$ as claimed.

By Lemma 3 the minimal value C_n^* of $\max_{1 \leq k \leq n-2} g(\Omega_k, \Omega_{k+1})$ taken over all sequences $(\Omega_1, \ldots, \Omega_{n-1})$ with $\Omega_k \in [1/4, 12]$ is a lower bound on $\widehat{C}_{2n+2}$. This completes the first step.

4.0.6. *

Optimal Residue Sequences

So now we are concerned with finding a residue sequence $(\Omega_1, \Omega_2, \ldots, \Omega_{n-1})$ with $\Omega_k \in [1/4, 12]$ such that $\max_{1 \leq k \leq n-2} g(\Omega_k, \Omega_{k+1})$ is minimized. More precisely, let $\Gamma : [1/4, 12]^{n-1} \longrightarrow I\!R$ with $\Gamma(\Omega_1, \Omega_2, \ldots, \Omega_{n-1}) = \max_{1 \leq k \leq n-2} g(\Omega_k, \Omega_{k+1})$ and $C_n^* = \inf_{\vec{\Omega} \in [1/4,12]^{n-1}} \Gamma(\vec{\Omega})$. As we observed above C_n^* is a lower bound on $\widehat{C}_{2n+2}$.

We first show that, for a fixed n, there is a special residue sequence $\vec{\Omega}^* = (\Omega_1^*, \Omega_2^*, \ldots, \Omega_{n-1}^*)$ such that $\Gamma(\vec{\Omega}^*) = C_n^*$ *and*, in addition, $g(\Omega_k^*, \Omega_{k+1}^*) = C_n^*$, for all $1 \leq k \leq n-2$.

Lemma 4 *There is a sequence* $(\Omega_1^*, \Omega_2^*, \ldots, \Omega_{n-1}^*) \in [1/4, 12]^{n-1}$ *such that* $g(\Omega_k^*, \Omega_{k+1}^*) = C_n^*$*, for all* $1 \leq k \leq n-2$.

Proof. F □

irst note that there is, indeed, a sequence $\vec{\Omega}^* \in [1/4, 12]^{n-1}$ with $\Gamma(\vec{\Omega}^*) = C_n^*$ since Γ is a continuous mapping on a compact domain and, therefore, assumes its minimum (and maximum).

Let $\mathcal{G}^*$ be the set of positive sequences $G = (g_1, g_2, \ldots, g_{n-2})$ of length $n-2$ such that $g_k = g(\Omega_k, \Omega_{k+1})$, for some sequence $(\Omega_1, \Omega_2, \ldots, \Omega_{n-1}) \in [1/4, 12]^{n-1}$ and $\max_{1 \leq k \leq n-2} g_k = C_n^*$. By the above argument $\mathcal{G}^*$ is not empty. Consider a sequence $G^* \in \mathcal{G}^*$ such that the number of elements g_k in G^* with $g_k = C_n^*$ is minimized. We claim that $G^* = (C_n^*, C_n^*, \ldots, C_n^*)$.

The proof is by contradiction. So assume that $G^* \neq (C_n^*, C_n^*, \ldots, C_n^*)$. Let $g_k \in G^*$ such that $g_k = C_n^*$ and either $g_{k-1} < g_k$ or $g_{k+1} < g_k$. Since G^* is not constant, such a g_k clearly exists. Consider the derivatives of $g_k = g(\Omega_k, \Omega_{k+1})$ w.r.t. Ω_k and Ω_{k+1}. We obtain

$$\frac{\partial g_k}{\partial \Omega_k} = \frac{\partial g}{\partial \Omega_k}(\Omega_k, \Omega_{k+1}) \quad = \quad 1 + \sqrt{\frac{2\left(1 + \frac{2}{\Omega_{k+1}}\right)}{\Omega_k + 2}} > 0 \tag{6}$$

and

$$\frac{\partial g_k}{\partial \Omega_{k+1}} = \frac{\partial g}{\partial \Omega_{k+1}}(\Omega_k, \Omega_{k+1}) \quad = \quad -\frac{4}{\Omega_{k+1}^2} - 2\frac{\sqrt{2(\Omega_k + 2)}}{\left(1 + \frac{2}{\Omega_{k+1}}\right)\Omega_{k+1}^2} < 0. \tag{7}$$

Hence, if $g_{k-1} < g_k$, then we can decrease Ω_k, which increases $g_{k-1} = g(\Omega_{k-1}, \Omega_k)$ and decreases $g_k = g(\Omega_k, \Omega_{k+1})$, such that g_k and $g_{k-1} < C_n^*$. In this way we obtain a new sequence G'. The number of elements $g'_i \in G'$ with $g'_i = C^*_n$ is one less than for G^*. Note that if g_k is the only element of G^* with $g_k = C^*_n$, then we have, in fact, decreased the value of $\max G^*$ in contradiction to our choice of $G^* \in \mathcal{G}^*$. Hence, there are less elements in G' that are equal to C^*_n than in G^* again in contradiction to the choice of G^*. Similarly, if $g_{k+1} < g_k$, then we can increase Ω_{k+1} which decreases $g_k = g(\Omega_k, \Omega_{k+1})$ and increases $g_{k+1} = g(\Omega_{k+1}, \Omega_{k+2})$ and the same argument applies. Hence, there is a sequence $(\Omega^*_1, \ldots, \Omega^*_{n-1})$ with $g(\Omega^*_k, \Omega^*_{k+1}) = C^*_n$, for all $1 \leq k \leq n-2$, as claimed.

In fact, the proof of Lemma 4 implies a stronger result.

Corollary 1 *All sequences* $\vec{\Omega} = (\Omega_1, \Omega_2, \ldots, \Omega_{n-1}) \in [1/4, 12]^{n-1}$ *with* $\Gamma(\vec{\Omega}) = C^*_n$ *satisfy* $g(\Omega_k, \Omega_{k+1}) = C^*_n$, *for* $1 \leq k \leq n-2$.

4.0.7. *

A Recurrence Equation

In the following let $(\Omega_1, \Omega_2, \ldots, \Omega_{n-1})$ be a sequence with $\Omega_k \in [1/4, 12]$ such that $g(\Omega_k, \Omega_{k+1}) = C^*_n$, for all $1 \leq k \leq n-2$. With the help of Lemma 4 we now can derive a recurrence equation for the sequence (Ω_k).

Lemma 5 *For all* $1 \leq k \leq n-2$,

$$\Omega_k = h(C^*_n, \Omega_{k+1}) = C^*_n - 2 + \frac{4}{\Omega_{k+1}} - 2\sqrt{2}\sqrt{\frac{(\Omega_{k+1}+2)(C^*_n - 2)}{\Omega_{k+1}}}. \qquad (8)$$

Proof. B □

y assumption the sequence (Ω_k) satisfies the equation

$$6 + \Omega_k + \frac{4}{\Omega_{k+1}} + 2\sqrt{2(\Omega_k + 2)\left(1 + \frac{2}{\Omega_{k+1}}\right)} = C^*_n,$$

for all $1 \leq k \leq n-1$. If we solve the above equation for Ω_k, then there are two possible solutions for Ω_k

$$\Omega_k = C^*_n - 2 + \frac{4}{\Omega_{k+1}} + 2\sqrt{2}\sqrt{\frac{(\Omega_{k+1}+2)(C^*_n - 2)}{\Omega_{k+1}}}$$

or

$$\Omega_k = C^*_n - 2 + \frac{4}{\Omega_{k+1}} - 2\sqrt{2}\sqrt{\frac{(\Omega_{k+1}+2)(C^*_n - 2)}{\Omega_{k+1}}}$$

In order to see that Ω_k equals the second solution we note that we only have to consider $\Omega_k \in [1/4, 12]$. Inequalities 6 and 7 imply that $C^*_n \geq g(1/4, 12) \geq 10$ and, hence,

$$\min_{\Omega_{k+1} \in [1/4, 12]} C^*_n - 2 + \frac{4}{\Omega_{k+1}} + 2\sqrt{2}\sqrt{\frac{(\Omega_{k+1}+2)(C^*_n - 2)}{\Omega_{k+1}}} \geq 16.9$$

which contradicts $\Omega_k \leq 12$. Therefore, Ω_k is given by Equation 8.

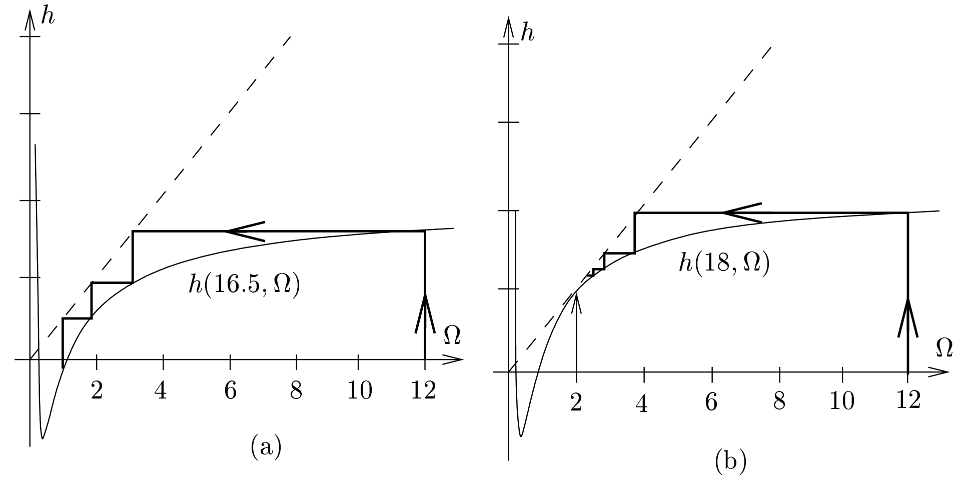

Figure 9: (a) For $\Omega > 1/4$, $h(16.5, \Omega)$ is completely below the diagonal of the first quadrant and after a few iterations Ω_k becomes negative. (b) The graph of $h(18, \Omega)$ touches the diagonal in the point $(2, 2)$ and an infinite number of positive values of Ω_k are possible.

4.0.8. *

Bounding the Number of Steps

Recurrence equation 8 limits the number of elements the sequence $(\Omega_1, \Omega_2, \ldots, \Omega_{n-1})$ can consist of for a given C_n^*. This can be seen by visualizing the dynamics of recurrence equation 8. In the following we refer to Figure **??**. Let C be a fixed value. Consider the graph $G_{h,C}$ of the function $h(C, \cdot)$. We obtain Ω_k if we start on the x-axis at Ω_{k+1} and go vertically up to $G_{h,C}$. At the intersection point $(\Omega_{k+1}, h(C, \Omega_{k+1})) = (\Omega_{k+1}, \Omega_k)$ of $G_{h,C}$ with the vertical line through Ω_{k+1} we continue horizontally until we intersect the diagonal of the first quadrant, in the point (Ω_k, Ω_k). At this point we again continue vertically until we intersect $G_{h,C}$ in (Ω_k, Ω_{k-1}) and so on (see Figure **??**).

We will show that $C = 18$ is the first value for which $G_{h,C}$ intersects the diagonal. Therefore, if $C < 18$, then Ω_k becomes negative or at least smaller than $1/4$ after some number of steps, say m. Since we require that $\Omega_k \geq 1/4$, for all $1 \leq k \leq n-1$, the length $n-1$ of the sequence (Ω_k) is at most $m-1$. In following we show that m is bounded by $36/(18-C)$.

We first show that Ω_k actually becomes negative or smaller than $1/4$ after a number of steps. To see this consider the roots of $h(C, \Omega_k)$ which are given by

$$z_1 = \frac{4}{C-10} - \frac{8\sqrt{2}}{\sqrt{C-2}(C-10)} \qquad \text{and} \qquad z_2 = \frac{4}{C-10} + \frac{8\sqrt{2}}{\sqrt{C-2}(C-10)}.$$

It is easy to see that $h(C, \Omega_k)$ is positive, for $\Omega_k \in [0, z_1)$ and $\Omega_k \in (z_2, \infty)$ and negative for $\Omega_k \in (z_1, z_2)$. Moreover, z_1 decreases as C increases, if $C \geq 10$, and, therefore, z_1 assumes its maximum for the minimal value of C. Hence,

$$z_1 \leq \lim_{C\to 10+} \frac{4}{C-10} - \frac{8\sqrt{2}}{\sqrt{C-2}(C-10)} \leq 1/4$$

and $\Omega_k \in [z_2, \infty)$, for all $1 \leq k \leq n-1$.

Lemma 6 *If* $10 \leq C < 18$*, then the maximum number* m *of elements* Ω_k *with* $\Omega_k \in [z_2, 12]$*, for all* $1 \leq k \leq m$*, and* $\Omega_k = h(C, \Omega_{k+1})$*, for all* $1 \leq k \leq m-1$*, is at most* $36/(18-C)$*.*

Proof. W □

e claim that

$$\Omega_k - \Omega_{k-1} \quad \geq \quad (18-C)/3, \tag{9}$$

for $\Omega_k, \Omega_{k-1} \in [z_2, \infty)$ and $10 \leq C < 18$. Since $z_2 \leq \Omega_1$, and $\Omega_k \leq 12$, this implies that

$$m \leq \frac{12-z_2}{\min_{1\leq k\leq n-1} \Omega_k - \Omega_{k-1}} \leq \frac{12-z_2}{(18-C)/3} \leq \frac{36}{18-C}$$

as claimed.

In order to see (9) we observe that by Lemma 5

$$\Omega_k - \Omega_{k-1} = \Omega_k - C + 2 - \frac{4}{\Omega_k} + 2\sqrt{2\frac{(\Omega_k+2)(C-2)}{\Omega_k}} \stackrel{def}{=} q(\Omega_k, C).$$

We consider the derivative q_C of q w.r.t. C.

$$\begin{aligned} q_C(\Omega_k, C) &= \sqrt{\frac{2(\Omega_k+2)}{\Omega_k(C-2)}} - 1 = \sqrt{2}\sqrt{\frac{2/\Omega_k+1}{(C-2)}} - 1 \ \leq\ \sqrt{2}\sqrt{\frac{2/z_2+1}{(C-2)}} - 1 \\ &= \sqrt{\frac{-8\sqrt{C-2}+4\sqrt{2}+\sqrt{C-2}\,C}{(\sqrt{C-2}+2\sqrt{2})\,(C-2)}} - 1 \ =\ -\sqrt{\frac{2}{C-2}} \ \leq\ -0.35. \end{aligned}$$

If $q(\Omega_k, 18) \geq 0$, then

$$\Omega_k - \Omega_{k-1} = q(\Omega_k, C) \geq q(\Omega_k, C) - q(\Omega_k, 18) = \int_C^{18} -q_C(\Omega_k, \gamma)\, d\gamma \geq 0.35(18-C)$$

which proves (9).

It remains to be shown that

$$q(\Omega_k, 18) = \Omega_k - 16 - \frac{4}{\Omega_k} + 2\sqrt{32\frac{\Omega_k+2}{\Omega_k}} \geq 0,$$

for $\Omega_k \in [z_2, 12)$. Since $\Omega_k - 16 - 4/\Omega_k < 0$, it suffices to show that

$$128\frac{\Omega_k+2}{\Omega_k} \geq \left(16 + \frac{4}{\Omega_k} - \Omega_k\right)^2.$$

If we multiply by Ω_k^2, the above inequality is equivalent to showing that the function

$$q_1(\Omega_k) = 128\,\Omega_k(\Omega_k+2) - (16\,\Omega_k + (2-\Omega_k)(2+\Omega_k))^2 = -\Omega_k^4 + 32\,\Omega_k^3 - 120\,\Omega_k^2 + 128\,\Omega_k - 16$$

is non-negative on $[z_2, 12)$. The function q_1 has three extrema, one at $11 - \sqrt{105}$, one at 2, and one at $11 + \sqrt{105}$. Obviously, the extremum at 2 is a minimum with $q_1(2) = 0$. Hence, we only need to check the boundary values $q_1(1/4) \approx 8.99 > 0$ and $q_1(12) = 18800 > 0$. Hence, q_1 is non-negative. This concludes the proof.

We are now in a position to finally prove Lemma 2 and, thus, Theorem 4′.

Proof. [□

Lemma 2] By Lemma 3 C_n^* is a lower bound for $\widehat{C}_{2n+2}$. By Lemma 4 there is a sequence $\vec{\Omega}^* = (\Omega_1^*, \Omega_2^*, \ldots, \Omega_{n-1}^*)$ with $\Omega_k^* \in [1/4, 12]$ such that $g(\Omega_k^*, \Omega_{k+1}^*) = C_n^*$, for all $1 \leq k \leq n-2$. Lemma 5 implies that $\vec{\Omega}^*$ satisfies the recurrence equation

$$\Omega_k^* = h(C_n^*, \Omega_{k+1}^*),$$

for all $1 \leq k \leq n-2$. By our previous considerations $\Omega_k^* \in [z_2, 12]$, for all $1 \leq k \leq n-1$. Hence, we can apply Lemma 6 and

$$n \leq \frac{36}{18 - C_n^*} \qquad \text{or} \qquad C_n^* \geq 18 - \frac{36}{n}$$

which concludes the proof.

Notice that an unbiased strategy can be viewed as a special case of a biased strategy; therefore, Theorem 4′ also implies a lower bound for unbiased strategies; however, the converse does not hold. In this sense Theorem 4′ is the stronger result. For other results about unbiased searching on the real line up to a given distance see [8,14].

Corollary 2 ([13]) *Let C^D be the competitive ratio for finding a target point on the real line under a given strategy X if the target is placed at a distance of at most D to s. Then, $C^D \geq 9 - O(1/\log D)$.*

Proof. T □

he proof is by contradiction. Assume that $C^D < 9 - 72/(\log D - 8)$. Then, $C_D^{av} < 9 - 72/(\log D - 8)$ since $C^D = \max_{1 \leq k \leq n-1}\{L_k, R_k\}$—a contradiction.

5. Conclusions

We have presented lower bounds for streets and generalized streets. In streets, we provide a lower bound of $\sqrt{2} - O(1/\sqrt{n})$ for the competitive ratio of any deterministic strategy that a robot may use to search in a rectilinear street if the coordinates of the target are given in advance to the robot. Here, n is the Euclidean distance from the start point to the target.

In $\mathcal{G}$-streets, we provide a simple example, that settles the competitive ratio of searching in orthogonal $\mathcal{G}$-streets w.r.t. the L_2-metric. We show that $\sqrt{82}$ is a lower bound which matches the competitive ratio of the best known algorithm. Secondly, we also investigate if it is an advantage for the robot if it is given the location of the target in advance. We show that there are polygons for every strategy that force the robot to walk at least nine times the length of the shortest path from s to t.

Our lower bounds are based on a new result about searching on the real line. Here, we show that the average competitive ratio of any strategy to search on the real line is at least $9 - O(1/\log D)$ if the target is placed anywhere within D to the origin.

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