Going Home Through an Unknown Street*

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Abstract

We consider the problem of a robot traversing an unknown polygon with the aid of standard visibility. The robot has to find a path from a starting point s to a target point g. We provide upper and lower bounds on the ratio of the distance traveled by the robot in comparison to the length of a shortest path. Since this ratio is unbounded for general polygons, we restrict ourselves to the well investigated class of polygons called *streets*. A *street* is a polygon where s and t are located on the polygon boundary and the part of the polygon boundary from s to g is weakly visible to the part from g to s and vice versa.

We consider two problems in this context. First we assume that the location of the target g is known to the robot. We prove a lower bound of $\sqrt{2}$ on the competitive ratio of any deterministic algorithm that solves this problem. This bound matches the competitive ratio for searches in a rectilinear polygon with an unknown target which implies that, for rectilinear streets, knowledge of the location of the destination provides no advantage for the robot. In addition, we also obtain a lower bound of 9 for the competitive ratio of searching in generalized streets with known target which closely matches the upper bound if the target is unknown.

Secondly, we consider a new strategy for searching in an arbitrarily oriented street where the location of g is unknown. We show that our strategy achieves a competitive ratio of $\sqrt{1+(1+\pi/4)^2}$ ($\sim$ 2.05) which significantly improves the best previously known ratio of $2\sqrt{1+1/\sqrt{2}}$ ($\sim$ 2.61).

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1 Introduction

One of the main problems in robotics is to find a path from the current location of the robot to a given target. While most of the work in this area has focussed on efficient algorithms for path planning if the robot is given a map of its environment in advance, a more natural and realistic setting is to assume that the robot has only a partial knowledge of its surroundings.

In this paper we assume that the robot is equipped with a vision system that provides a visibility map of its *local* environment. Based on this information the robot has to find a path to a given target that is located somewhere within the scene. The search of the robot can be viewed as an on-line problem since it discovers its surroundings as it travels. Hence, one way to analyze the quality of a search strategy is to use the framework of competitive analysis as introduced by Sleator and Tarjan [12]. A search strategy is called *c-competitive* if the path traveled by the robot to find the target is a most c times longer than a shortest path. c is called the *competitive ratio* of the strategy.

Since there is no strategy with a competitive ratio of $o(n)$ for scenes with arbitrary obstacles having a total of n vertices [2], the on-line search problem has been studied previously in various contexts where the geometry of the obstacles is restricted. Papadimitrou and Yannakakis were the first to consider the case of traversing an unknown scene with rectangular obstacles in search of a target whose location is known [11]. They show a lower bound of $\Omega(\sqrt{n})$ for the competitive ratio of any strategy. Later Blum, Raghavan, and Schieber provided a strategy that achieves this bound [2]. If the aspect ratio or the length of the longest side of the rectangles are bounded, better strategies are possible [3, 10].

Kleinberg studies the problem of a robot searching inside a simple polygon for an unknown goal located on the boundary of the polygon [8]. He introduces the notion of *essential cuts* inside a polygon of which there may be considerably fewer than polygon vertices and gives an $O(m)$-competitive strategy for orthogonal polygons with m essential cuts.

Klein introduced the notion of a *street* which al-

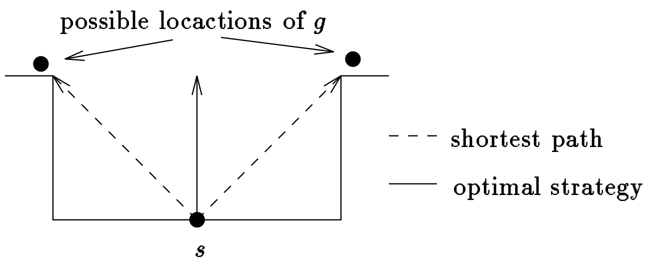

Figure 1: A lower bound for searching in rectilinear streets.

lowed for the first time a search strategy with a constant competitive ratio even though the location of the target is unknown [7]. In a street the starting point s and the target t are located on the boundary of the polygon and the two polygonal chains from s to t are mutually weakly visible. Klein presents a strategy for searching in streets and gives an upper bound on its competitive ratio of $1 + 3/2\pi$ (~ 5.71). The analysis was recently improved to $\pi/2 + \sqrt{1 + \pi^2/4}$ (~ 4.44) by Icking [6]. Though Klein's strategy performs well in practice—he reports that no example had been found for which his strategy performs worse than 1.8—the strategy and its analysis are both quite involved and no better competitive ratio could be shown until recently,when Kleinberg presented a new approach. His strategy for searching in streets allows to prove a competitive ratio of $2\sqrt{2}$ with a very simple analysis [8]. Moreover, for rectilinear streets Kleinberg shows that his strategy achieves a competitive ratio of $\sqrt{2}$ which is optimal due to the trivial example shown in Figure 1. Here, if a strategy moves to the left or right before seeing g, then g can be placed on the opposite side, thus forcing the robot to travel more than $\sqrt{2}$ times the diagonal. Curiously enough, this is the only known lower bound even for arbitrarily oriented streets.

Finally, a more general class of polygons, called $\mathcal{G}$-streets, has been introduced by Datta and Icking that allows search strategies with a competitive ratio of 9.06 [4]. All the these strategies fall into the category of Unknown Destination Searches (UDS) in which the location of the goal is unknown.

One natural source of information for the robot are the coordinates of the target. The first problem we consider is a lower bound for strategies for Known Destination Searches (KDS) in a street where the location of the goal is given in advance to the robot. In this case the example of Figure 1 obviously no longer provides a lower bound. We prove that even in orthogonal streets a $\sqrt{2}$-competitive ratio is optimal as well, thus providing the first non-trivial lower bound for searching in streets. This result is different from the general search problem as considered by Papadimitrou and Yannakakis in which knowledge of the destination improves the competitive ratio.

Secondly we consider a new strategy for searching in arbitrarily oriented streets. We achieve a competitive ratio of $\sqrt{1 + (1 + \pi/4)^2}$ (~ 2.05), providing a significant improvement over previous strategies and the best performance guarantee for searching strategies in streets known so far.

The paper is organized as follows. In Section 2 we introduce the basic geometric concepts necessary for the rest of the paper. In particular, we give a precise definition of a street. In Section 3 we show that any deterministic search algorithm for orthogonal streets that knows the location of the target can be forced to travel $\sqrt{2} - O(1/\sqrt{n})$ times the distance of a shortest path to the target where n is the number of vertices of the polygon. Finally, Section 4 deals with a new strategy to search in streets and its analysis.

2 Definitions and Assumptions

We consider a simple polygon P in the plane with n vertices and a robot inside P which is located at a start point s on the boundary of P. The robot has to find a path from s to the target point g. The search of the robot is aided by simple vision (i.e. we assume that the robot knows the visibility polygon of its current location). Furthermore, the robot retains all the information seen so far (in memory) and knows its starting and current position. We are, in particular, concerned with a special class of polygons called *streets* first introduced by Klein [7].

Definition 2.1 [7] *Let P be a simple polygon with two distinguished vertices, s and g, and let L and R denote the clockwise and counterclockwise, resp., oriented boundary chains leading from s to g. If L and R are mutually weakly visible, i.e. if each point of L sees at least one point of R and vice versa, then (P, s, g) is called a* street.

Definition 2.2 *In the class of Known Destination Search (KDS) problems, a robot searches a simple rectilinear polygon, starting from s on the boundary of the polygon, for a target point g on the boundary of P with known location.*

We denote the L_2-distance between two points p_1 and p_2 by $d(p_1, p_2)$ and the L_2-norm of a point p by

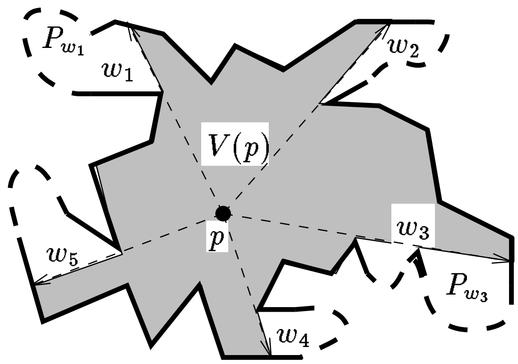

Figure 2: The the visibility polygon $V(p)$ of p with windows $w_1, \ldots, w_5$.

$||p||$.

Definition 2.3 *Let P be a street with start point s and target g. If p is a point of P, then the* visibility polygon *of p is the set of all points in P that are seen by p. It is denoted by $V(p)$. A* window *of $V(p)$ is an edge of $V(p)$ that does not belong to the boundary of P (see Figure 2).*

A window w splits P into a number of subpolygons $P_1, \ldots, P_k$ one of which contains $V(p)$. We denote the union of the subpolygons that do not contain $V(p)$ by P_w.

3 A $\sqrt{2}$-competitive lower bound

We construct a family of polygons which are $(\sqrt{2}-\epsilon)$-competitive for KDS, for any $\epsilon > 0$. First, we define some widgets which will be used in the general construction.

Definition 3.1 *An* eared rectangle *is a rectangle two units wide and one unit tall. The center of the base is the* entry point *and on the top left and right corners there are two small alleys (ears) attached to it (see Figure 3). One of the alleys is* connecting, *the other is* a dead alley.

Definition 3.2 *The* aspect ratio *A of a general polygon is defined as the ratio between the smallest and the longest edge of the polygon. Thus $A \leq 1$.*

Lemma 3.1 *An eared rectangle may be traversed from the entry point to the connecting alley at a $(\sqrt{2}-\epsilon)$-competitive ratio, with $\epsilon = O(A)$, which is optimal.*

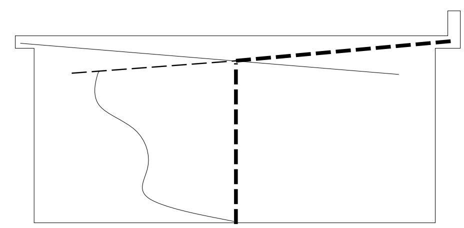

Figure 3: Eared Rectangle, with walk inside.

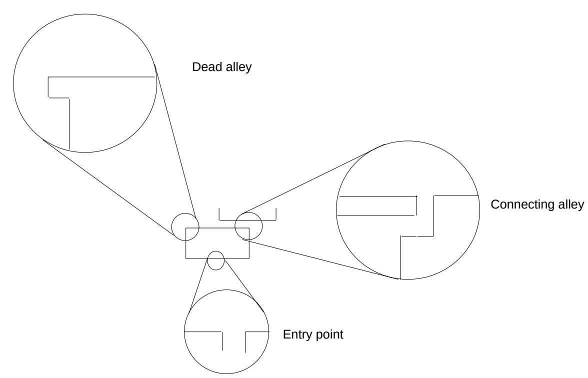

Figure 4: Interconnecting Eared Rectangles.

Proof: First we show that a $\sqrt{2}$-competitive ratio is attainable. The robot walks up the middle of the rectangle, until it sees the top boundaries of both alleys. At this time the robot can see into either alley and determine which one is open, and proceed to walk in this direction (see bold dashed lines in Figure 3). The length of the trajectory is $1-\tan\theta+1/\cos\theta$, where θ is the angle of the line between the extreme upper and the closer lower end point of the alleys. Notice that θ can be made arbitrarily small by means of reducing the height of the alley. Thus, this strategy gives a walk of length arbitrarily close to $\sup_{\theta\to 0}\{1+1/\cos\theta-\tan\theta\} = 2$. The optimal walk is of length $\sqrt{2}$ for a competitive ratio of $\sqrt{2}-\epsilon$ where $\epsilon = (1/\sqrt{2})(1+\tan\theta-1/\cos\theta) = O(A)$.

This strategy is optimal as well. We use an adversary argument to show this. The adversary simply opens the first alley to be looked into by the robot, and closes the other alley. Clearly the alley opened is always in the opposite half of the rectangle in which the robot is currently located. (see curvy path plus dashed line in Figure 3). A simple application of the triangle inequality shows that the path in bold is shorter, and thus has a better competitive ratio.

□

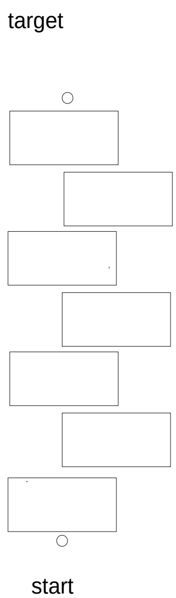

Figure 5: Walk the Middle Policy

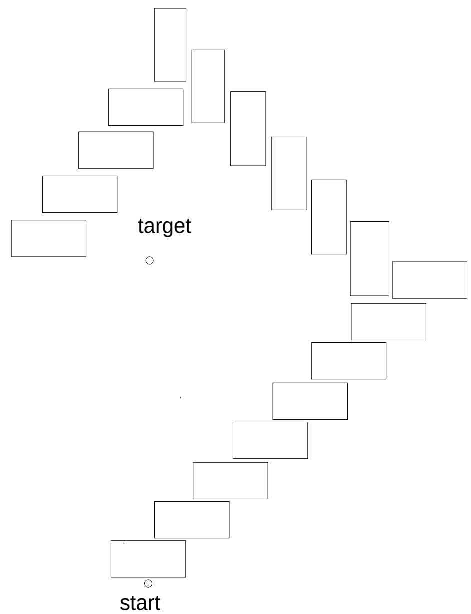

Figure 6: Always to the Right Policy

Eared rectangles can be interconnected to create paths. Figure 4 shows the details of such rectangles.

Theorem 1 *There exists a street with n vertices which can be searched with an optimal competitive ratio of* $\sqrt{2} - O(1/\sqrt{n})$.

Proof: As proved by Kleinberg [8], there exists a $\sqrt{2}$-competitive strategy for UDS which can be applied in a straightforward way to the KDS problem and gives a strategy of the same competitive ratio for all polygons in the KDS problem.

What remains to be shown is that this competitive ratio is optimal. We assume that the target is a some distance directly above the start point as shown in Figure 5. To prove a lower bound of $\sqrt{2}$ we first consider two extreme cases of interconnecting eared rectangles, namely the *Walk the Middle Policy* and the *Always to the Right Policy*.

If the algorithm uses a strategy such as the one proposed in Lemma 3.1, the construction of Figure 5 shows an example of a polygon with a competitive ratio of $(2n+1)/(\sqrt{2}n+1)$, where n is the number of rectangles between the start point and the target.

Thus, an algorithm needs to deviate from the *Walk the Middle Policy*. In this case, the adversary presents the algorithm with an eared rectangle and it opens and closes the alleys according to the strategy proposed in Lemma 3.1. If we assume that the algorithm always meets the line of sight in the left half, then the adversary consistently opens the right alley (see Figure 6). This creates a staircase moving to the right. Notice that the L_1 distance from the current robot position to the target is always within one unit of the L_1 distance from the start point to the target. That is, the adversary has forced the algorithm to move at a worse than $\sqrt{2}$ competitive ratio, but the target is no closer than before.

When the current connecting alley is now horizontally aligned with the target, the adversary moves one unit closer to the target (we assume that the algorithm also moves optimally in this part, since it knows the position of the target) and proceeds to construct a new staircase. This results in a spiraling set of staircases converging to the start point. The spiral is of length quadratic in n (see [1]) and, thus, the competitive ratio is $O((2n^2+n)/(\sqrt{2}n^2+n))$ which goes to $\sqrt{2}$ as n goes to infinity.

Having analyzed these extreme cases, we now consider a *Wavering Policy* in which the algorithm neither walks up the middle, nor consistently slants either way (see Figure 7). The *Walk the Middle Policy* and *Always to the Right Policy* can be viewed as extreme instances of the *Wavering Policy*.

In the case of a wavering algorithm, the adversary maintains the strategy described above. Every time the

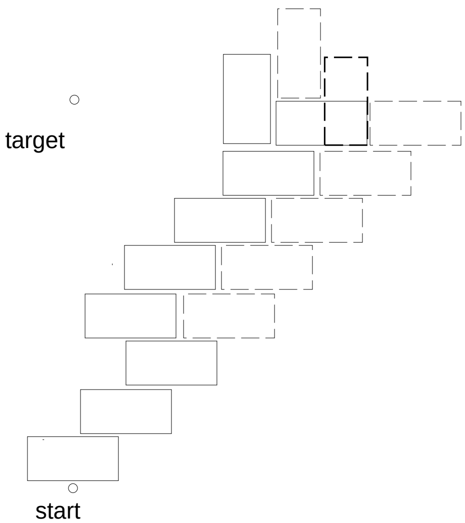

Figure 7: Wavering Policy

algorithm deviates from the *Always to the Right Policy*, the adversary moves to the left. As a consequence, the L_1 distance to the target is reduced by two units, while the competitive ratio remains above $\sqrt{2}$.

From the point of view of the algorithm a left turn, is equivalent to a "jump" from one level of the spiral to a level on the spiral associated to a start point two units closer to the target (see Figure 7, with the polygon in solid lines, and the older staircase in dashed lines).

Assume $n = 2m$ is even. Let k be the number of turns to the left deviating from the *Always to the Right Policy*. Without loss of generality let $k \leq m$, since the the case $k > m$ can be seen as a deviation from the symmetric *Always to the Left Policy*. Furthermore, assume that the algorithm jumps at staircases $a_1, a_2, \ldots, a_k$, where the staircases are numbered, starting from 1, in the order they are traversed. Then, the total length of the path traversed by the algorithm is

$$2 \sum_{0 \leq j < a_1} (n - 2j) + 2 +$$
$$2 \sum_{a_1 \leq j < a_2} (n - 2a_1 - 2j) + 2 +$$
$$2 \sum_{a_2 \leq j < a_3} (n - 2a_1 - 2a_2 - 2j) + 2 + \cdots$$

where each sum represents the length of a segment of a spiral staircase in between jumps.

Lemma 3.2 *Consider two strategies for walking up the staircase. Strategy A turns left in staircases* $\{a_i\}_{1 \leq i \leq k}$, *and Strategy B turns left in the staircases* $\{b_i\}_{1 \leq i \leq k}$, *such that* $b_i = a_i - 1$, *for all* i *with* $a_i > 1$, *and* $b_i = 1$ *otherwise. Then strategy B has a better competitive ratio than strategy A.*

Proof: Since $a_i \geq b_i$ it follows that the summation above is, term by term, larger for strategy A than for strategy B, from which the claim follows. □

Thus, setting $a_i = 1$, for all i, is optimal. Let $n = 2m$. If the algorithm jumps or turns left k times, then we have

$$\text{Length of shortened spiral} = n + \sum_{i=0}^{m-k} (n - 2k - 2i) = n + (m-k)(m-k+1)$$

Length of optimal walk =
$$\sqrt{2}\,(n + (m-k)(m-k+1)) + n - 2k$$

Distance traversed by the algorithm =
$$2\,(n + (m-k)(m-k+1)) + n - 2k$$

$$\text{Competitive ratio} = \sqrt{2} - \xi \text{ where}$$

$$\xi = \frac{2(m-k)(\sqrt{2}-1)}{\sqrt{2}\,(n + (m-k)(m-k+1)) + n - 2k}.$$

To improve its competitive ratio, the algorithm can select the optimal value of k for all given m that maximizes ξ. As proven in Appendix A, $k = m - \sqrt{2m}$ maximizes ξ to $O(1/\sqrt{m})$. Since each eared rectangle is traversed at a $\sqrt{2} - O(A)$ competitive ratio, we have that the adversary strategy described above forces any algorithm into a $\sqrt{2} - O(A) - O(1/\sqrt{n})$ competitive ratio, which in the limit is $\sqrt{2}$.

As it can be seen, regardless of the policy, a $\sqrt{2}$ inefficiency factor is necessarily introduced, even in the case where the robot knows where it is going, but is ignorant of the terrain in which is moving. □

Datta and Icking [4] introduced the notion of $\mathcal{G}$-streets and showed for the UDS problem a competitive ratio of 9.06.

Definition 3.3 **([4])** *A simple polygon in the plane is called a* generalized street *if for every boundary point* $p \in L \cup R$, *there exists a horizontal chord with end points in* L *and* R *and from which* p *is weakly visible.*

Datta and Icking proved a lower bound by building a "rake" polygon which is traversed at a 9-competitive ratio on the limit (see Figure 8).

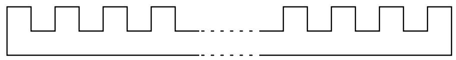

Figure 8: A “rake” polygon.

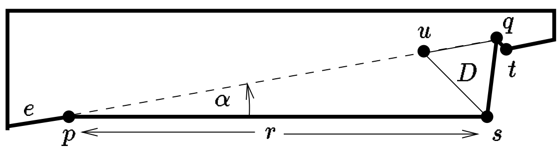

Figure 9: A lower bound example for Kleinberg's strategy.

For the KDS problem, as in the case of eared rectangles, a single rake polygon does not suffice to obtain a lower bound on the competitive ratio of 9.

A similar strategy to the one presented results in a 9-competitive ratio for this problem.

Corollary 1 *There exists a family of orthogonal $\mathcal{G}$-streets with 9-competitive ratio, which is close to optimal.*

Proof: Consider a target at a distance n of the start point, and placed directly above it on the vertical direction. Then a set of n^2 rakes placed one above the other, each of height $1/n$, gives the desired lower bound. □

4 Traversing a Street

In this section we present a new strategy to traverse a street from the starting point s to the target point g. Our strategy closely follows the approach of Kleinberg [8] in order to deal with the simple cases which his strategy handles optimally but deviates in the more complicated cases.

The competitive ratio of Kleinberg's strategy is shown to be $2\sqrt{2}$ (~ 2.83) in his paper but a tighter analysis—also mentioned in his paper—yields an upper bound of $2\sqrt{1+1/\sqrt{2}}$ (~ 2.61). Before we describe our approach we show that this bound is tight for his strategy. Figure 9 shows an example of a polygon where his bound is achieved asymptotically.

Here Kleinberg's strategy follows the diagonal D to point u where the chain to the left of point p is visible to the robot and then moves to the right to point q. In order to see that a competitive ratio of $2\sqrt{1+1/\sqrt{2}}$ is achieved let $d = \sqrt{2}+1$ and q be chosen to have the coordinates $(q_1, q_2) = (1/\sqrt{d^2+1}, d/\sqrt{d^2+1})$ where we assume s to be the origin. If p has distance r to s and we observe that $\sin\alpha = q_1/(r+q_2)$, then $u = (u_1, u_2) = (r\sin\alpha/(\sin\alpha - 1), r\sin\alpha/(\sin\alpha-1))$. Hence, the total distance traveled by a robot following Kleinberg's strategy is $\geq \sqrt{2}u_2 + u_1 + q_1$. If we take into account that $u_1 = u_2 = q_2 - O(1/r)$, then the distance traveled by the robot is $q_1 + (\sqrt{2}+1)q_2 - O(1/r) = 1/\sqrt{d^2+1} + d^2/\sqrt{d^2+1} - O(1/r) = \sqrt{d^2+1} - O(1/r)$ which tends to $2\sqrt{1+1/\sqrt{2}}$ as r approaches $+\infty$. By keeping edge e collinear with q and moving g closer and closer to q it can be seen that the claimed ratio can be achieved arbitrarily closely.

Our strategy can be shown to have a competitive ratio of at most $\sqrt{1+(1+\pi/4)^2}$ (~ 2.05) but in contrast to Kleinberg's analysis, the analysis of our strategy is not tight. Before we describe the strategy we need a few definitions and observations.

As observed by Kleinberg the shortest path $\mathcal{P}_{st}$ from s to g consists of a number of line segments that touch reflex vertices of P. The general strategy we follow is to start at a reflex vertex v of P that belongs to $\mathcal{P}_{st}$ and to identify another reflex vertex v' of $\mathcal{P}_{st}$ that is closer to g by traveling further on. If the robot has identified v', then it moves to it and starts the search anew. A move from one reflex vertex of P on $\mathcal{P}_{st}$ to another closer to g is called a *step*.

Recall that a window is an edge of the boundary of the visibility polygon $V(p)$ of p that does not belong to the boundary of P.

All windows are collinear with p. The end point of a window w that is closer to p is called the *entrance point* of w. We assume that a window w has the orientation of the ray from p to entrance point of w. We say a window w is a *left window* if P_w is locally to the left of w w.r.t. the given orientation of w. A *right window* is defined similarly.

If the robot has traveled along the path $\mathcal{P}$, then we assume that the robot knows the part of P that can be seen from $\mathcal{P}$, i.e. the robot has seen maintains the polygon $V(\mathcal{P}) = \bigcup_{p\in\mathcal{P}} V(p)$. We say a window w of $V(p)$ is a *true window* w.r.t. $\mathcal{P}$ if it is also a window of $V(\mathcal{P})$.

We have the following lemma about true windows.

Lemma 4.1 *If w is a right (left) window of $V(p)$ and the boundary of P_w belongs to L (R), then w is not a true window.*

We say two windows w_1 and w_2 are *clockwise consecutive* if the clockwise oriented polygonal chain of

$V(p)$ between w_1 and w_2 does not contain a window different from w_1 and w_2. *Counterclockwise consecutive* is defined analogously.

Lemma 4.2 *All windows that belong to L (R) are clockwise (counterclockwise) consecutive in $V(p)$.*

True windows are called *consecutive* if there is no true window that is between them. An immediate corollary of Lemmas 4.1 and 4.2 is that true left and true right windows are consecutive.

Corollary 4.3 *If w_0 is the window that is intersected by $\mathcal{P}$ the first time, then all true left (right) windows are clockwise (counterclockwise) consecutive from w_0 in $V(p)$.*

Because of Corollary 4.3 there is a clockwise-most true left entrance point from w_0 which we denote by p^+ and a counterclockwise-most true right entrance point of $V(p)$ which we denote by p^- if $V(p)$ contains both true left and right windows. The point p^+ is called the *left extreme entrance point* and p^- the *right extreme entrance point* of $V(p)$.

Now assume the robot starts at s and travels towards its target. We consider five cases:

Case 1 g is visible to the robot.
The robot moves to g on a straight line.

Case 2 There is no true left window (right window).
The robot moves to p^- (p^+).

Case 3 The angle at the location of the robot between p^+ and p^- is greater than or equal to $\pi/2$.
We apply Algorithm *Move-in-Quadrant* as described below until we are able to decide which of p^+ and p^- is part of a shortest path from s to g.

If none of the above cases apply, then p^+ and p^- are defined and the angle at p between p^+ and p^- is less than $\pi/2$. The robot chooses a direction such that p^+ is to the left of the direction and p^- to the right. It travels following the direction until one of the above or one of the following two cases occurs.

Case 4 A new point p^+ or p^- appears and p, p^+ and p^- are collinear.
The robot moves along the line through p, p^+, and p^- to the closer point of p^+ and p^- (see Figure 10).

Case 5 The angle at the location of the robot between p^+ and p^- equals $\pi/2$.
We apply Algorithm *Move-in-Quadrant* as described below until we are able to decide which of p^+ and p^- is part of a shortest path from s to g.

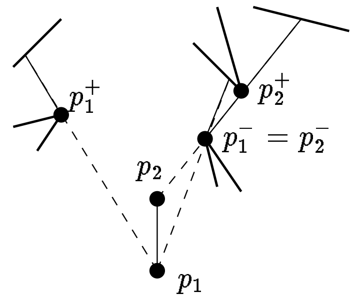

Figure 10: As the robot moves to p_2 the left extreme entrance point "jumps" from p_1^+ to p_2^+ and the robot moves directly to p_1^-.

The *orthogonal projection* p' of a point p onto a line segment l is defined as the point of l that is closest to p. If the line segment joining p with p' is orthogonal to l, then p' is a *non-degenerate* orthogonal projection.

Algorithm Move-in-Quadrant
Input: A point p_0 in P such that the angle at p_0 of p_0^+ and p_0^- of $V(p_0)$ is $\geq \pi/2$;
$i := 0$;
while p_i^+ and p_i^- of $V(p_i)$ are defined **do**
(1) Move to the orthogonal projection p_{i+1} of p_0 onto the line segment l_i from p_i^+ to p_i^-;
Compute the points p_{i+1}^+ and p_{i+1}^- of $V(p_{i+1})$;
$i := i + 1$;
end while;

The correctness of the algorithm follows from the following lemma.

Lemma 4.4 *If the robot has reached the line segment l_i, then one of p_i^+ or p_i^- is not an extreme entrance point of $V(p_{i+1})$ anymore.*

For the analysis consider the Cases 1, 2, and 4 first. In the Cases 1 and 2 the robot moves directly to the next point on a shortest path from s to g, hence, the competitive ratio is 1. If Case 4 occurs before Algorithm *Move-in-Quadrant* is invoked, then the angle between the line segment from the robot to p_i^+ or p_i^- and the traveling direction is less or equal $\pi/2$ which implies that if the robot moves directly to p_i^+ or p_i^-, then the competitive ratio is bounded by $\sqrt{2}$.

4.1 Analysis of the Algorithm *Move-in-Quadrant*

If during the movement in Step (1) one of the Cases 1 or 4 occurs, then the robot moves immediately to g or the closer of the points p^+ or p^-. Nevertheless, for the analysis we assume that the robot first moves to the line segment determined by the old points p^+ and p^- and then to the closer one of the two.

In the following we assume that the Algorithm *Move-in-Quadrant* has stopped after k iterations.

Lemma 4.5 *During the Algorithm Move-in-Quadrant the shortest path from s to g goes through either p_i^+ or p_i^-, for all $0 \leq i \leq k$.*

Because of Lemma 4.5 it suffices to bound the ratio of the length of the path traveled by the robot to the distance between p_0 and p_k^+ or p_k^-—whichever is detected as a part of the shortest path from s to g.

The situation we analyze is displayed in Figure 11. We introduce a coordinate system where p_0 is the origin and the x-axis passes through p_0^-. Note that since the angle at p_0 between between p_0^+ and p_0^- is greater than or equal to $\pi/2$, there is no reflex vertex of P in the first quadrant of the introduced coordinate system that is visible to p_0. Assume that we have arrived at point p_i and move to point p_{i+1} in the next iteration. We make a few simple observations about the locations of p_i^+, p_i^-, and the line segment l_i from p_i^+ to p_i^-.

Lemma 4.6 *The point p_i^+ belongs to the second quadrant and the point p_i^- belongs to the fourth quadrant, for all $0 \leq i \leq k$.*

Lemma 4.7 *The line segments l_i and l_{i+1} do not intersect in the first quadrant.*

Since the line segment l_i intersects the first, second, and fourth quadrant, the orthogonal projection of p_0 onto l_i is non-degenerate.

In order to simplify the analysis we consider the line segment l_i' from the intersection point of l_i with the y-axis to the intersection point of l_{i+1} with the x-axis as shown Figure 11.

The line segment l_i' is located between l_i and l_{i+1}. If we consider the path $\mathcal{P}_i'$ from p_0 to p_i that visits the orthogonal projections of p_0 onto the line segments l_j and l_j' in order, for $0 \leq j \leq i$, then the length of $\mathcal{P}_i'$ is obviously greater than or equal to the length of $\mathcal{P}_i$. Furthermore, $\mathcal{P}_i$ and $\mathcal{P}_i'$ share the same start and end point. Hence, for the simplicity of exposition we assume in the following that p_i^+ is located on the y-axis, p_i^- on the x-axis, and either $p_i^+ = p_{i+1}^+$ or $p_i^- = p_{i+1}^-$.

Let L_i be the length of the path $\mathcal{P}_i$ traveled by the robot to reach p_i; let α_i^- be the angle between the line segment $\overline{p_0p_i}$ from p_0 to p_i and the x-axis and d_i^- the distance between p_0 and p_i^-. Similarly, let α_i^+ be

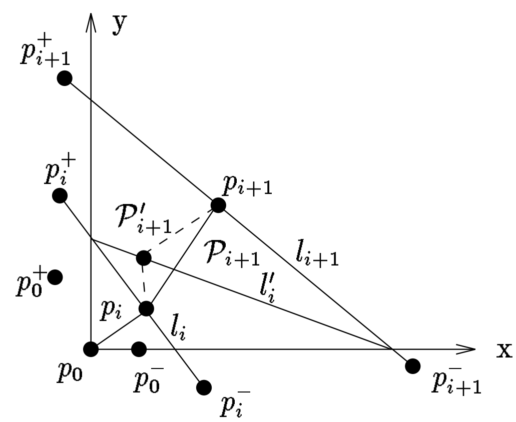

Figure 11: Introducing a new segment between l_i and l_{i+1}.

the angle between $\overline{p_0p_i}$ and the y-axis and d_i^+ the distance between p_0 and p_i^+. We define the angle α_i as $\min\{\alpha_i^+, \alpha_i^-\}$ and the distance d_i as $\min\{d_i^+, d_i^-\}$.

Our approach to analyze our strategy is based on the idea of a potential function. Each point p_i is assigned a potential Q_i which is defined as $Q_i = \alpha_i d_i$. It is our aim to show that $L_i + Q_i \leq (1 + \pi/4)d_i$, for all $0 \leq i \leq k$.

So suppose the robot has reached the point p_i and d_i is equal to the distance between p_0 and p_i^- and $L_i \leq (1+\pi/4-\alpha_i)d_i$. Note that since $d_i = d(p_0, p_i^-)$, the line segment l_i has a slope greater than or equal to $\pi/4$ and p_i is below the diagonal of the first quadrant. Hence, α_i is the angle between the line segment $\overline{p_0p_i}$ and the x-axis. For simplicity of description we assume that the distance from p_0 to p_i^+ is 1 and, therefore, $d_i = \tan\alpha_i$ as can be seen in Figure 12.

The robot moves now from p_i to p_{i+1}. We distinguish three cases.

Case 1 The line segment l_{i+1} is steeper than the line segment l_i.
Hence, $d_{i+1} = d_i$. Note that p_{i+1} is on the circle C_i with center at $c_i = (d_i/2, 0)$ and radius $d_i/2$ (see Figure 13a). The arc a_i of C_i from p_i to p_{i+1} has length $2(\alpha_i - \alpha_{i+1})d_i/2 = (\alpha_i - \alpha_{i+1})d_i$ since the angle between p_i and p_{i+1} at c_i is $2(\alpha_i - \alpha_{i+1})$. Clearly, the line segment $\overline{p_ip_{i+1}}$ is shorter than the arc a_i. Hence,

$$\begin{aligned} L_{i+1} &= L_i + d(p_i, p_{i+1}) \\ &\leq (1 + \frac{\pi}{4} - \alpha_i)d_i + (\alpha_i - \alpha_{i+1})d_i \\ &= (1 + \frac{\pi}{4} - \alpha_{i+1})d_{i+1} \end{aligned}$$

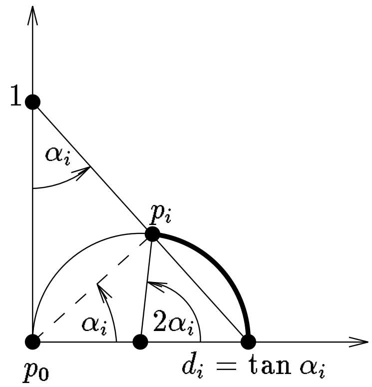

Figure 12: If $d(p_0, p_i^+) = 1$, then $d(p_0, p_i^-) = \tan \alpha_i$ and the length of the thick circular arc is $\tan \alpha_i \cdot \alpha_i$.

Case 2 The line segment l_{i+1} is steeper than $\pi/4$ but less steep than the line segment l_i (see Figure 13b). Hence, $d_{i+1} = \tan \alpha_{i+1}$. Note that p_{i+1} is on the circle C_i' with center at $c_i' = (0, 1/2)$ and radius $1/2$. The arc a_i' of C_i' from p_i to p_{i+1} has length $2(\alpha_{i+1} - \alpha_i)1/2 = \alpha_{i+1} - \alpha_i$ since the angle between p_i and p_{i+1} at c_i' is $2(\alpha_i - \alpha_{i+1})$. Clearly, the line segment $\overline{p_i p_{i+1}}$ is shorter than the arc a_i. Hence,

$$\begin{aligned} L_{i+1} &= L_i + d(p_i, p_{i+1}) \\ &\leq (1 + \frac{\pi}{4} - \alpha_i)d_i + (\alpha_{i+1} - \alpha_i) \end{aligned}$$

We want to show that

$$(1 + \frac{\pi}{4} - \alpha_i)d_i + (\alpha_{i+1} - \alpha_i) \quad \leq \quad (1 + \frac{\pi}{4} - \alpha_{i+1})d_{i+1} \tag{1}$$

or

$$\begin{aligned} 1 + \frac{\pi}{4} &\geq \frac{\alpha_{i+1}d_{i+1} - \alpha_i d_i + \alpha_{i+1} - \alpha_i}{d_{i+1} - d_i} \\ &= \frac{\alpha_{i+1}(1 + \tan \alpha_{i+1}) - \alpha_i(1 + \tan \alpha_i)}{\tan \alpha_{i+1} - \tan \alpha_i} \end{aligned}$$

with $0 \leq \alpha_i \leq \alpha_{i+1} \leq \pi/4$. If define $\beta_i = \alpha_{i+1} - \alpha_i$ and $f(\alpha_i, \beta_i) = (\beta_i + (\alpha_i + \beta_i)\tan(\alpha_i + \beta_i) - \alpha_i \tan \alpha_i)/(\tan(\alpha_i + \beta_i) - \tan \alpha_i)$, then we want to prove that $f(\alpha, \beta) \leq \pi/4$, for all $(\alpha, \beta) \in \Delta = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq \pi/4\}$. As a first step we show that

$$\frac{\partial f}{\partial \alpha}(\alpha, \beta) = \frac{\sin \beta + \beta(\cos(2\alpha + \beta) - \sin(2\alpha + \beta))}{\sin \beta} \geq 0,$$

for all $(\alpha, \beta) \in \Delta$. To see this consider

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial f}{\partial \alpha}(\alpha, \beta) \sin \beta\right) = -2\beta(\sin(2\alpha + \beta) + \cos(2\alpha + \beta))$$

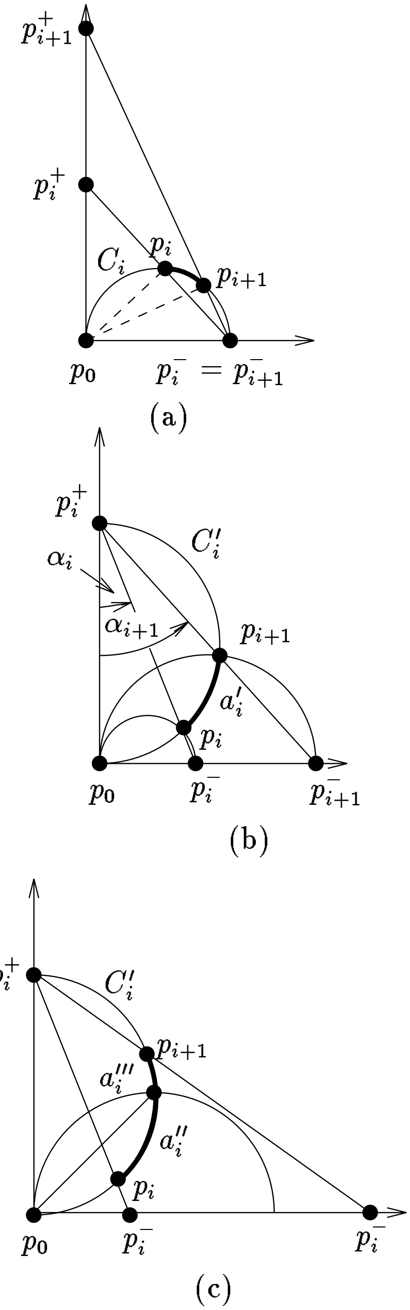

Figure 13: Cases 1, 2, and 3 if the robot moves from p_i to p_{i+}.

which is less or equal 0, for all $(\alpha,\beta)\in\Delta$, since $\sin(2\alpha+\beta)\geq 0$ and $\sin(2\alpha+\beta)\geq -\cos(2\alpha+\beta)$, for $0\leq 2\alpha+\beta\leq 3\pi/4$. Hence,

$$\begin{aligned}\min_{(\alpha,\beta)\in\Delta}\frac{\partial f}{\partial\alpha}(\alpha,\beta) &= \min_{\beta\in[0,\pi/4]}\frac{\partial f}{\partial\alpha}\left(\frac{\pi}{4}-\beta,\beta\right)\\ &= \min_{\beta\in[0,\pi/4]}\frac{\sin\beta+\beta(\sin\beta-\cos\beta)}{\sin\beta}\\ &\geq 0.\end{aligned}$$

Therefore, f is monotone in α and $\max_{(\alpha,\beta)\in\Delta} f(\alpha,\beta) = \max_{\beta\in[0,\pi/4]} f(\pi/4-\beta,\beta)$. If $g(\beta)=f(\pi/4-\beta,\beta)$, then $dg/d\beta = (2\beta-\sin\beta)/(\cos 2\beta - 1)\leq 0$ and, therefore,

$$\begin{aligned}\max_{(\alpha,\beta)\in\Delta} f(\alpha,\beta) &= \max_{\beta\in[0,\pi/4]} g(\beta)\\ &= \lim_{\beta\to 0}\frac{\beta+\frac{\pi}{4}\tan(\frac{\pi}{4})-(\frac{\pi}{4}-\beta)\tan(\frac{\pi}{4}-\beta)}{\tan(\frac{\pi}{4})-\tan(\frac{\pi}{4}-\beta)}\\ &= \lim_{\beta\to 0}\frac{\beta}{\sin(2\beta)}(1+\cos(2\beta))+\frac{\pi}{4} = 1+\frac{\pi}{4}\end{aligned}$$

Case 3 The line segment l_{i+1} is less steep than $\pi/4$ (see Figure 13c).

Hence, d_{i+1} is now the distance from p_0 to $p_{i+1}^+ = p_i^+$ which is 1 by our assumption. Furthermore, α_{i+1} is the angle between $\overline{p_0p_{i+1}}$ and the y-axis. If α'_{i+1} is the angle between $\overline{p_0p_{i+1}}$ and the x-axis, then $\alpha_{i+1}+\alpha'_{i+1}=\pi/2$.

Note that p_{i+1} is on the circle C'_i with center at $c'_i=(0,1/2)$ and radius $1/2$. The arc a'_i of C_i from p_i to p_{i+1} has length $2(\alpha'_{i+1}-\alpha_i)1/2 = \alpha'_{i+1}-\alpha_i$. Again, the line segment $\overline{p_ip_{i+1}}$ is shorter than the arc a'_i. And as above we obtain

$$\begin{aligned}L_{i+1} &= L_i + d(p_i,p_{i+1})\\ &\leq (1+\frac{\pi}{4}-\alpha_i)d_i + (\alpha'_{i+1}-\alpha_i)\end{aligned}$$

We split a'_i into two arcs a''_i and a'''_i where a''_i is the arc from p_i to the diagonal of the first quadrant and a'''_i is the arc from the diagonal of the first quadrant to p_{i+1}. The arc a''_i is paid for by the increase $d_{i+1}-d_i$ while the arc a'''_i just reduces the potential. More precisely, we have

$$\begin{aligned}&(1+\frac{\pi}{4}-\alpha_i)d_i+(\alpha'_{i+1}-\alpha_i)\\ &= (1+\frac{\pi}{4}-\alpha_i)d_i+(\frac{\pi}{4}-\alpha_i)+(\alpha'_{i+1}-\frac{\pi}{4})\\ &\leq (1+\frac{\pi}{4}-\frac{\pi}{4})\cdot 1+(\frac{\pi}{4}-\alpha_{i+1})\\ &= (1+\frac{\pi}{4}-\alpha_{i+1})d_{i+1}\end{aligned}$$

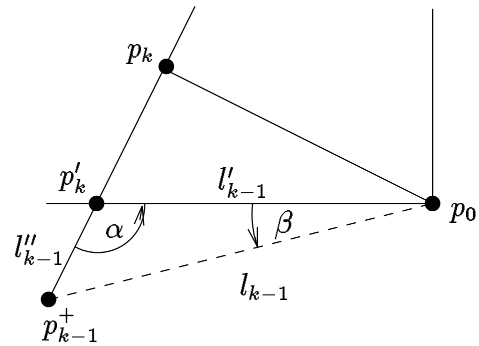

Figure 14: Bounding the competitive ratio in Case 3.

where the last inequality follows from Inequality (1) if we set $\alpha_{i+1}=\pi/4$. This proves the claim.

In fact we have shown the following lemma.

Lemma 4.8 *For all* $0\leq i\leq k$,

$$1+\frac{\pi}{4}\geq\max\left\{\frac{L_i+d(p_i,p_i^+)}{d(p_0,p_i^+)},\frac{L_i+d(p_i,p_i^-)}{d(p_0,p_i^-)}\right\}$$

4.2 Analysis of Cases 3 and 5

In order to obtain the final competitive ratio for one step we have to take into account that the robot has to move to either p^+_{k-1} or p^-_{k-1}. First consider Case 3. If p_k^- is undefined, then p^+_{k-1} belongs to the shortest path from s to g. Lemma 4.8 gives a tight bound on the maximum distance the robot travels in order to reach p'_k in Figure 14.

Let l''_{k-1} be the line segment between p^+_{k-1} and p'_k and α the angle between l'_{k-1} and l''_{k-1}. The length of l_{k-1} grows monotonously with α if the lengths of l'_{k-1} of l''_{k-1} are fixed. Hence, the maximum ratio is assumed for the minimum angle α which is $\alpha=\pi/2$. If we set l_{k-1} to have length 1, then the length of l'_{k-1} is $\cos\beta$ and the length of l''_{k-1} is $\sin\beta$. Hence, the maximum distance traveled by the robot from p_0 to p^+_{k-1} is bounded by

$$\max_{0\leq\beta\leq\pi/2}\sin\beta+c\cos\beta$$

or

$$\max_{0\leq x\leq 1}\sqrt{1-x^2}+cx.$$

where $c=1+\pi/4$. This maximum is achieved for $x=c/\sqrt{c^2+1}$ and yields a value of $\sqrt{c^2+1}$. The same analysis applies if p_k^+ is undefined.

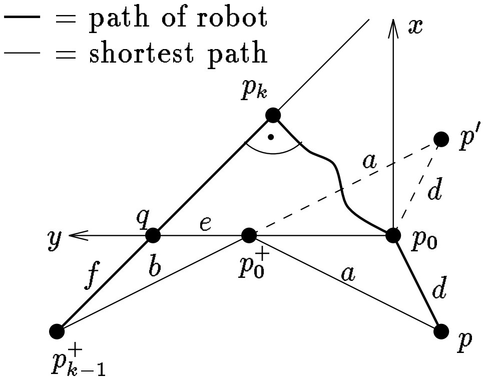

Figure 15: Computing the competitive ratio in Case 5.

Now we consider Case 5. This case turns out to be somewhat more complicated than the previous one. Let p be the point where the robot started its search. We denote the position of the robot at time t by $p(t)$ with left entrance point $p^+(t)$ and right entrance point $p^-(t)$. After traveling some distance d the robot encounters a point $p_0 = p(t_0)$ where the angle between $p^+(t_0)$ and $p^-(t_0)$ is exactly $\pi/2$. Note that the angle between $p^+(t)$ and $p^-(t)$ at the robot position is a continuous, monotonously increasing function if the robot moves on a ray. At p_0 the robot invokes the Algorithm *Move-in-Quadrant*. Let S be the coordinate system with origin p_0 and p_0^+ on the y-axis and p_0^- on the x-axis both in the first quadrant. Note that the x- and the y-coordinate of p in S are non-positive.

Suppose that p_k^- is undefined. Then, a shortest path from s to g visits p (by the induction hypothesis), p_0^+, and p_{k-1}^+. Hence, we have the situation displayed in Figure 15.

The length of the shortest path from p to p_{k-1}^+ is at least the sum of the distance a from p to p_0^+ and the distance b from p_0^+ to p_{k-1}^+. We first consider for which point p_0^+ the sum $a+b$ is minimized given p, p_0, and p_{k-1}^+. In order to compute this point p_0^+ we reflect p at the y-axis to a point p' and note that the distance between p' and p_0^+ equals a (see Figure 15). Hence, $a+b$ is minimized if p_0^+ is located on the line from p_{k-1}^+ to p'. Furthermore, the distance d from p to p_0 which is traveled by the robot is maximized if p is located on the x-axis given fixed p_0, p_0^+, and a.

Now we consider the maximum of $e+f$ given p_{k-1}^+ where the line segment l from q to p_{k-1}^+ must form an angle of less or equal $3\pi/2$ with the y-axis. As can be seen by applying the triangle inequality it is obtained if l is orthogonal to the y-axis (see Figure 16).

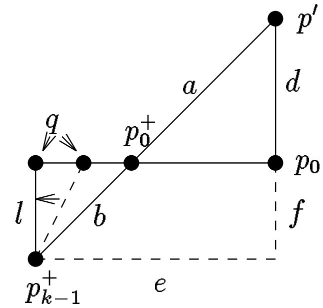

Figure 16: A triangle with a right angle can be formed.

So assume q has the same y-coordinate as p_{k-1}^+. If we translate the line segment l such that q coincides with p_0 and also translate the line segment from p_0 to q such that q coincides with p_{k-1}^+ (see Figure 16), then we obtain a triangle whose sides have length e, $d+f$ and $a+b$ with a right angle in its right hand corner. The previous analysis applies and

$$\sqrt{1+c^2} \geq \frac{d+f+ce}{a+b}$$

with $c = 1+\pi/4$ as claimed. We have shown the following theorem.

Theorem 2 *If P is a street with start point s and target point g, then there is a strategy for a robot with access to the local visibility map of its surroundings to travel from s to g on a path that is at most $\sqrt{1+(1+\pi/4)^2}$ times longer than the shortest possible route.*

5 Conclusions

We consider two problems in this paper. First we provide a lower bound of $\sqrt{2}-O(1/\sqrt{n})$ for the competitive ratio of any deterministic strategy that a robot may use to search in a rectilinear street if the coordinates of the target are given in advance to the robot. This implies that knowledge of the location of the target does not provide any advantage even for searching in rectilinear streets. We further show a similar result for $\mathcal{G}$-streets.

Secondly, we present a new strategy to search in arbitrarily oriented streets. We show a performance guarantee of $\sqrt{1+(1+\pi/4)^2}\,(\sim 2.05)$ for our strategy.

Unfortunately, the gap between the upper and lower bounds for searching in streets is still quite large and it seems that new ideas are needed to narrow it down.

A Analysis of ξ

To maximize ξ we can equivalently find the value of k that minimizes c/ξ, for c a constant.

$$\begin{aligned}\xi &= \frac{2(\sqrt{2}-1)(m-k)}{\sqrt{2}\,[2m+(m-k)(m-k+1)]+2m-2k} \\ &= \frac{\sqrt{2}(\sqrt{2}-1)}{2m/(m-k)+m-k+1+\sqrt{2}} \quad \Rightarrow \\ \frac{c}{\xi} &= \frac{2m}{m-k}+m-k+1+\sqrt{2}\end{aligned}$$

Let

$$\begin{aligned}d(m,k) &= \frac{2m}{m-k}+m-k+1+\sqrt{2} \\ \frac{\partial}{\partial k}d(m,k) &= 2\frac{m}{(m-k)^2}-1 \\ \frac{\partial^2}{(\partial k)^2}d(m,k) &= 4\frac{m}{(m-k)^3}\end{aligned}$$

With the critical points of $d(m,k)$ at $k_{1,2}=m\pm\sqrt{2m}$. Since $k\leq m$ we need only to study $k_1=m-\sqrt{2m}$. Consider $d(m,\cdot)$ as a function of k. We see that the second derivative is positive for all $k<m$, and in particular at $k+1$. This implies that $d(m,\cdot)$ is minimized at k_1.

Thus for a fixed m, the best competitive ratio is attained at $k_1=m-\sqrt{2m}$, namely $\sqrt{2}-\xi$ with $\xi=\sqrt{2}(\sqrt{2}-1)/(2\sqrt{2m}+1+\sqrt{2})$. Notice that ξ goes to zero as m goes to infinity.

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