Drawing $K_{2,n}$: A Lower Bound*

Therese Biedl† Timothy M. Chan† Alejandro López-Ortiz†

1 Introduction

In graph drawing [1], the main objective is to obtain a representation of a graph in the plane under some aesthetic or functional criteria. For the purposes of visualization and chip layout, one would like to devise planar embeddings in a rectangular grid that have both small area and small aspect ratio. Here, the area of an embedding is defined as WH and the aspect ratio as $\max\{W/H, H/W\}$, if the minimum enclosing axis-parallel rectangle has width W and height H.

It is known that some graphs on n vertices require $\Omega(n^2)$ area for any planar embedding. A related question is whether there are graphs for which a constant aspect ratio can only be achieved at the expense of non-optimal area usage. Steve Wismath at the 2001 Graph Drawing Symposium [2] conjectured that a graph containing $K_{2,n}$ has non-constant aspect ratio for all optimal-area, planar embeddings.

In this note we show that indeed this is the case, provided that "embeddings" are taken to mean *straight-line* drawings, where vertices are mapped to grid points and edges are mapped to noncrossing straight line segments.

2 Preliminaries

$K_{2,n}$ is the complete bipartite graph with two vertices (say a and b) in one class and n vertices (say $v_1, \ldots, v_n$) in the other class, and all possible edges between them. Figure 1 shows an embedding of $K_{2,n}$ in an $n \times 2$ grid. Since clearly $K_{2,n}$ needs $\Omega(n)$ grid points, this drawing has asymptotically optimal area. However, the aspect ratio is $\Theta(n)$. A natural question is hence: What is the smallest area of a drawing of $K_{2,n}$ that has a constant aspect ratio? Or more specifically, is it possible to draw $K_{2,n}$ in an $O(\sqrt{n}) \times O(\sqrt{n})$-grid? (It is not difficult to see that the graph $K_{1,n}$ can be drawn that way; in contrast, $K_{3,n}$ is not planar for $n \geq 3$.)

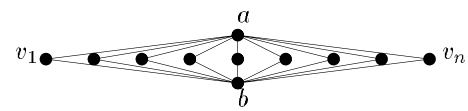

Figure 1: A drawing of $K_{2,n}$ of area $O(n)$.

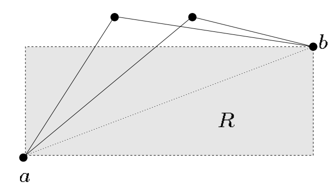

Figure 2: Each row outside R cannot contain two vertices without introducing crossings.

We first state an easy lemma for counting grid points inside rectangles (one possible self-contained proof is given for completeness):

Lemma 2.1 *A $w \times h$ rectangle (of arbitrary slopes) can contain at most $O((w+1)(h+1))$ grid points.*

Proof: Suppose there are k grid points inside the rectangle. Draw disks of radius 1/2 around each such point. These disks are disjoint and their union has area $(\pi/4)k$, but since the union is contained in a $(w+1) \times (h+1)$ rectangle, $(\pi/4)k \leq (w+1)(h+1)$. □

3 The Proof

Consider a planar straight-line drawing of $K_{2,n}$ in a $W \times H$ grid. We will upper-bound the number of vertices n in terms of W and H, thereby showing that W or H must be large. Without loss of generality, assume $W \geq H$.

Let R be the minimum axis-parallel rectangle enclosing a and b. Let ℓ be the line through a and b, denote by L the distance between a and b, and denote by D the distance of ℓ to an opposite corner of

*Research supported by NSERC. The authors would like to thank the participants of the Algorithms Problem Session at University of Waterloo for helpful input.

†School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1, email: {biedl,tmchan,alopez-o}@uwaterloo.ca

R. Observe that $L = O(W)$ and $D = O(H)$. In the sequel, we will only count vertices v_j's in the upper halfplane of ℓ, as vertices in the lower halfplane can be dealt with similarly. (See Figure 3 (left).)

Any horizontal line above R can contain at most one vertex (see Figure 2), and similarly, any vertical line left of R can contain at most one vertex. Thus, at most $W + H = O(W)$ vertices can be drawn outside R. We therefore focus our attention now on counting vertices inside R.

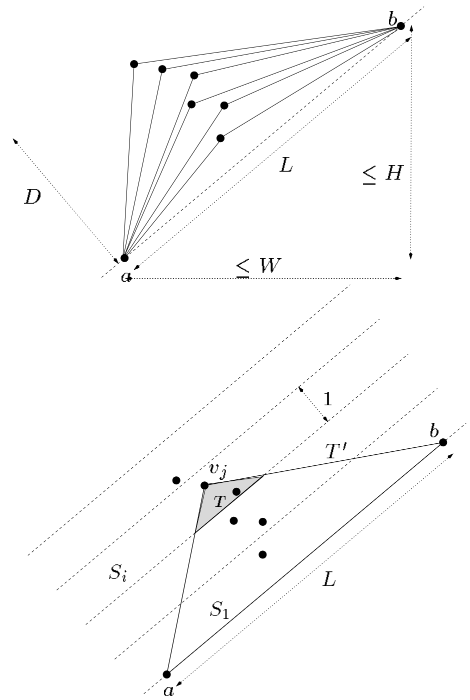

Figure 3: Proof of the claim.

Form $\lceil D \rceil$ strips $S_1, \ldots, S_{\lceil D \rceil}$ of width 1, where S_i contains all points above ℓ whose distance to ℓ lies in the interval $(i-1, i]$ (see Figure 3 (right)).

Claim 1 *Strip S_i contains at most $O(L/i+1)$ vertices inside R.*

Proof: Assume that $S_i \cap R$ contains at least one vertex, and let v_j be the farthest such vertex from the line ℓ. Let T' be the triangle $\triangle a v_j b$ and let T be the intersection of T' with S_i (the triangle T is shaded in Figure 3).

Note that no vertex within S_i can be outside T, because otherwise its line towards a or b would cross an edge of T'. So the number of vertices within S_i is bounded by the number of grid points inside T. The height of T is at most 1, and the width of T is at most L/i, because T is similar to T' (which has width L) and has at most $1/i$ times its height. Since $v_j \in R$, the triangle T is obtuse and is therefore contained in a $1 \times (L/i)$ rectangle. By Lemma 2.1, the number of grid points inside T is $O(L/i+1)$, which proves the claim. $\square$

The number of vertices inside R, in all strips together, is at most a constant times

$$\sum_{i=1}^{\lceil D \rceil} \left(\frac{L}{i} + 1 \right) \leq L(1+\ln \lceil D \rceil)+\lceil D \rceil = O(W \log H).$$

We conclude that the overall number of vertices n is bounded by $O(W \log H)$.

Theorem 1 *Every planar straight-line drawing of $K_{2,n}$ in a $W \times H$ grid with $W \geq H$ satisfies $W \log H = \Omega(n)$.*

In particular, one dimension W must exceed $\Omega(n/\log n)$ since $W = \Omega(n/\log H) = \Omega(n/\log n)$. So, if we have a drawing of $K_{2,n}$ with aspect ratio $O(1)$, then $H = \Theta(W) = \Omega(n/\log n)$ and the area is at least $\Omega(n^2/\log^2 n)$.

Furthermore, if a drawing has optimal area $WH = O(n)$, then by division, $H/\log H = O(1)$, i.e., $H = O(1)$, so one dimension W must be $\Omega(n)$.

4 Open problems

The main arguments in our proof (modulo the simple counting trick for vertices outside R) actually hold for any embedding where every pair of vertices is of distance at least 1. It is interesting to see if properties of the integer lattice would allow us to eliminate the extra $\log H$ factor in Theorem 1.

Another question that remains open is if the lower bounds shown hold for planar *polyline* drawings, where edges may bend at grid points.

References

[1] G. Di Battista, P. Eades, R. Tamassia, and I. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs.* Prentice-Hall, 1998.

[2] S. Felsner, G. Liotta, and S. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. In *Proc. Graph Drawing (GD 2001)*, Lect. Notes in Comput. Sci., vol. 2265, pages 328–342, 2001.